Section 15.1 - Double Integrals over Rectangles

- Review the Definite Integral:

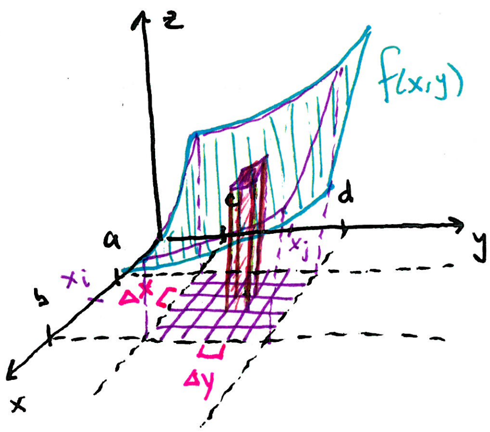
Sum of Areas of Rectangles: $\sum_{i=1}^{n} \underbrace{f(x_i)}_{\text{height}} \underbrace{\Delta x}_{\text{width}}$

Exact Area under f on [a,b] $= \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

★ Watch Rieman Sum Demo on Website

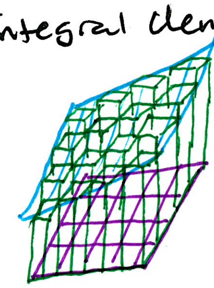
- Volumes and double Integrals:

Compute the volume under a Surface in a similar way - Sum of volumes of rectangular prisms



Visual ★ See Double Integral demo on Website

Approximate Volume under f(x,y)



- Volume of a rectangular cylinder: Area base × height $= \overbrace{(\Delta x \cdot \Delta y)}^{\Delta A} \times f(x_i, x_j)$

- Approximate Volume under f(x,y): $V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, x_j)\,\Delta A$

- Exact Volume under f(x,y): $V = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\,\Delta A$

The double Integral of f over R $= [a,b]\times[b,c] = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

$$\boxed{\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\,\Delta A}$$

Warning: $\iint_R f(x,y)\,dA$ is Volume only if $f(x,y) \ge 0$ on all of R!

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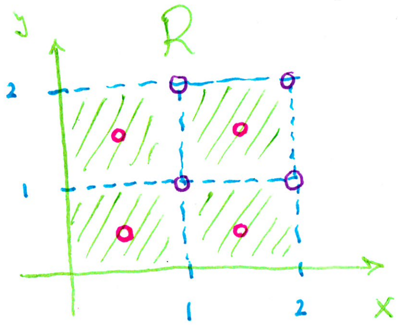
Example Estimate the volume of the solid that lies above the square $R=[0,2]\times[0,2]$ and below $z=16-x^2-2y^2$ by dividing R into 4 equal squares and choosing the upper right corner of each square for taking the height of the rectangular prism. Compare this approximation to the midpoint approximation.

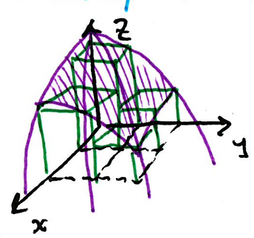


$$V \approx 1\cdot f(1,1)+1\cdot f(1,2)+1\cdot f(2,1)+1\cdot f(2,2)$$

$$=(16-1-2)+(16-1-8)+(16-4-2)+(16-4-8)$$

$$=\boxed{34 \text{ units}^3}$$

$\Delta A=\Delta x\Delta y=1\cdot 1=1$



$$V \approx 1\cdot f(0.5,0.5)+1f(0.5,1.5)+1\cdot f(1.5,0.5)+1\cdot f(1.5,1.5)$$

$$=\left(16-\tfrac{1}{4}-\tfrac{1}{2}\right)+\left(16-\tfrac{1}{4}-\tfrac{9}{2}\right)+\left(16-\tfrac{9}{4}-\tfrac{1}{2}\right)+\left(16-\tfrac{9}{4}-\tfrac{9}{2}\right)$$

$$=\boxed{49 \text{ units}^3}$$

- Average Value of $f(x)$ on $[a,b]$: $f_{ave}=\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ ← rectangle with height f_{ave} its area $=\int_a^b f(x)\,dx$

- Average Value of $f(x,y)$ on R: $\dfrac{1}{(\text{Area } R)}\displaystyle\iint_R f(x,y)\,dA = f_{ave}$ ← rectangular prism with height f_{ave} its volume $=\displaystyle\iint_R f(x,y)\,dA$

- Properties of Double Integrals:

① $\displaystyle\iint_R [f(x,y)+g(x,y)]\,dA=\iint_R f(x,y)\,dA+\iint_R g(x,y)\,dA$

② $\displaystyle\iint_R c\cdot f(x,y)\,dA = c\cdot\iint_R f(x,y)\,dA$

③ If $f(x,y)\ge g(x,y)$ for all $(x,y)\in R$ then

$$\iint_R f(x,y)\,dA \ge \iint_R g(x,y)\,dA$$

Warning the reverse is not always True!

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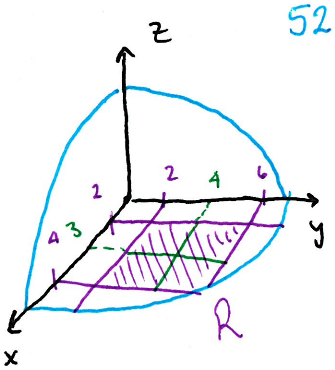
- Extra Examples

\#7 Let V be the volume of the solid under $f(x,y)=\sqrt{52-x^2-y^2}$ and above the rectangle given by $2 \le x \le 4$, $2 \le y \le 6$. Use $x=3$, $y=4$ to divide R into subrectangles. Without computing the Riemann sums with the lower left corner (L) and the upper right corner (R) arrange V, L, R in increasing order.



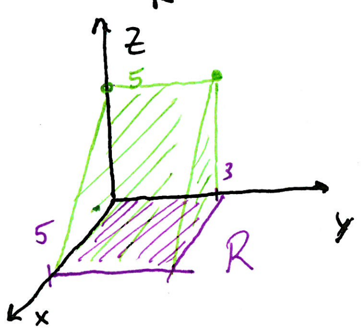
$52 = x^2+y^2+z^2$ is a sphere of radius $\sqrt{52}$ at the origin

Since $f(x,y)$ is decreasing as x, y increase we have

$$\boxed{R < V < L}$$

\# 12 Evaluate the double integral by identifying it as the volume of a solid,

$\iint_R (5-x)\,dA$ where $R=[0,5]\times[0,3]$.



$\iint_R (5-x)\,dA$ is the volume of a Triangular prism

$$\iint_R (5-x)\,dA = \frac{1}{2}(5\times 5)\times 3 = \boxed{\frac{75}{2}\text{ units}^3}$$

\#17 If f is a constant function $f(x,y)=K$ and $R=[a,b]\times[c,d]$

Show that $\iint_R K\,dA = K(b-a)(d-c)$

$$\iint_R K\,dA = \lim_{n,m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} K\cdot\Delta x\cdot\Delta y \quad \text{where } \Delta x = \frac{b-a}{n} \quad \Delta y = \frac{d-c}{m}$$

$$= K\cdot \lim_{n,m\to\infty} \sum_{i=1}^{n}\left(\frac{b-a}{n}\right)\cdot\sum_{j=1}^{m}\left(\frac{d-c}{m}\right)$$

$$= K\cdot \lim_{n,m\to\infty} n\cdot\left(\frac{b-a}{n}\right) m\left(\frac{d-c}{m}\right) = \boxed{K\cdot(b-a)(d-c)}$$