Section 14.8 – Lagrange Multipliers

Recall 14.7: Finding max/min value of $z=f(x,y)$ on a closed bounded set of $\mathbb{R}^2$

1. Find Critical points of f on D
2. Find extreme values of f on boundary of D
3. Test points, Compare values

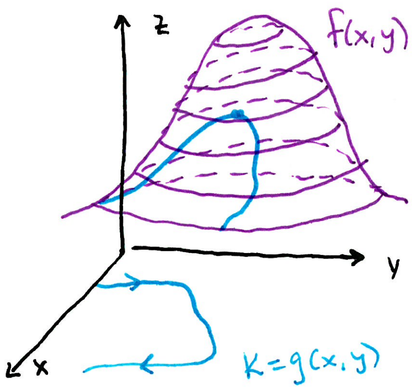
Now find the max/min value of $z=f(x,y)$ given some constraint on x and y

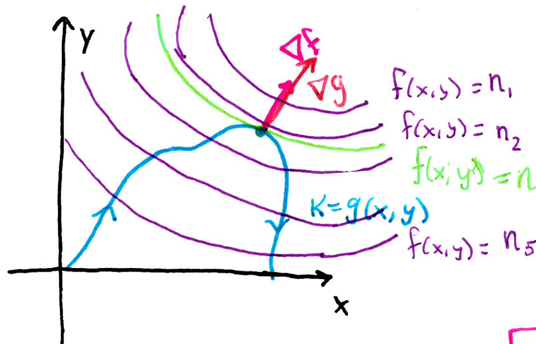
★ We'll look at the constaint on x and y when given by:

An equation: $g(x,y)=K$ Ex. f = Surface Area $K=g$ = volume constraint

- Methods to find max/min:

  1. Constraint: $y=g(x) \Rightarrow z=f(x,g(x))$ function of 1 variable find max/min by finding/Testing Critical numbers.
  2. Constraint: $K=g(x,y)$ curve in xy-plane





Extreme Values

When level curve of $f(x,y)$ touches curve $g(x,y)=K$

$\Rightarrow$ Common tangent line

$\Rightarrow$ parallel gradients

$\boxed{\nabla f = \lambda \nabla g}$ for a Scalar λ

- Method of Lagrange Multipliers: λ is called the Lagrange Multiplier

  To find max/min values of $f(x_1,x_2,\dots,x_n)$ subject to $g(x_1,x_2,\dots,x_n)=K$

  1. Find all values of $x_1,\dots,x_n$ and λ Satisfying $\nabla f = \lambda \nabla g$ and $g(x_1,\dots,x_n)=K$

     ↳ Provided max/min values exist and $\nabla g \neq 0$ on $g(x_1,\dots,x_n)=K$

  2. Evaluate f on all points (x,y,z) found in ① to find max/min

★ Good strategy for ① find $x_1,x_2,\dots,x_n$ in terms of λ then solve for λ

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Example Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2+y^2=1$.

① $f_x = \lambda g_x$ ② $f_y = \lambda g_y$ ③ $g(x,y)=1$ where $g(x,y) = x^2+y^2$

$2x = \lambda(2x)$ $\quad 4y = \lambda 2y$ $\quad x^2+y^2=1$

$x=0 / \lambda = 1$ $\quad y=0/\lambda=2$ $\quad$ Points: $(\pm 1, 0)$, $(0, \pm 1)$

$y=\pm 1 / y=0$ $\quad x=\pm 1 / x = 0$

$\boxed{f(\pm 1, 0) = 1 \text{ min}}$ $\quad \boxed{f(0,\pm 1) = 2 \text{ max}}$

Example Find the points on the sphere $x^2+y^2+z^2=4$ that are closest and farthest from the point $(3,1,-1)$.

$f(x,y,z) = d^2 = (x-3)^2 + (y-1)^2 + (z+1)^2$ $\quad g(x,y,z) = x^2+y^2+z^2 = 4$

① $2(x-3) = \lambda(2x)$ ② $2(y-1) = \lambda(2y)$ ③ $2(z+1) = \lambda(2z)$ ④ $g(x,y,z)=4$

$x = \frac{3}{1-\lambda}$ $\quad y = \frac{1}{1-\lambda}$ $\quad z = \frac{-1}{1-\lambda}$ Sub into ④

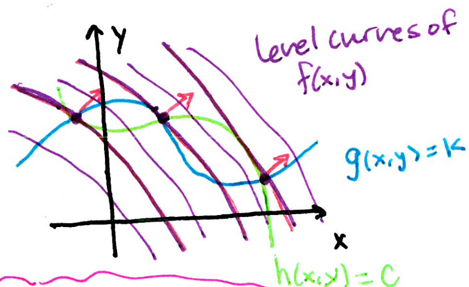
$4 = \left(\frac{3}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{-1}{1-\lambda}\right)^2 \Rightarrow 4(1-\lambda)^2 = 11 \Rightarrow \lambda = 1 \pm \sqrt{11}/2$

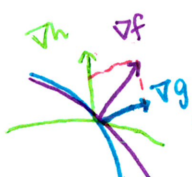
Points: $\left(\mp \frac{6}{\sqrt{11}}, \mp \frac{2}{\sqrt{11}}, \pm \frac{2}{\sqrt{11}}\right)$

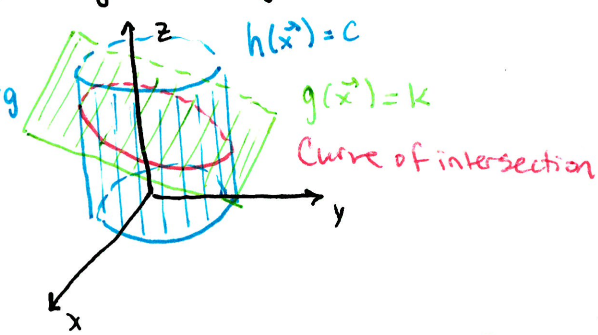
$\boxed{\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right) \text{ farthest}}$ $\quad \boxed{\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \text{ closest}}$

because opposite signs x, y, z

- <u>Subject to Two Constraints</u>: $f(\vec{x})$ subject to $g(\vec{x}) = k$ and $h(\vec{x}) = c$







May be Trivial in this case if a finite number of intersection points

★ Need gradient on the curve of intersection to be parallel to ∇f

$\Rightarrow \nabla f$ must be in the plane determined by ∇g and ∇h

$$\boxed{\nabla f = \lambda \nabla g + \mu \nabla h}$$

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MVC

Example Maximize $f(x,y,z) = x+2y+3z$ on the curve of intersection $x-y+z=1$ and $x^2+y^2=1$. $g(x,y,z) = x-y+z$ $h(x,y,z) = x^2+y^2$

① $1 = \lambda(1) + \mu(2x)$ ⇒ $x = -1/\mu$

② $2 = \lambda(-1) + \mu(2y)$ ⇒ $y = 5/2\mu$

③ $3 = \lambda(1) + \mu(0)$ ⇒ $\lambda = 3$

④ $x-y+z=1$

⑤ $x^2+y^2=1$

Sub x,y into ⑤: $\left(\frac{-1}{\mu}\right)^2 + \left(\frac{5}{2\mu}\right)^2 = 1$

$\mu = \pm \sqrt{29}/2$

$x = \mp 2/\sqrt{29}$

$y = \pm 5/\sqrt{29}$

Sub into ④

$z = 1 \pm 7/\sqrt{29}$

$f\left(\mp\frac{2}{\sqrt{29}}, \frac{\pm 5}{\sqrt{29}}, 1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$

$\boxed{\text{Max is } 3+\sqrt{29}}$

#20 Find extreme values of $f(x,y) = 2x^2+3y^2-4x-5$ on $x^2+y^2 \le 16$.

① Critical points: $\vec{0} = \nabla f = \langle 4x-4, 6y \rangle$ @ $(1,0)$ $f(1,0) = -7$

② Extreme Values on Boundary: Lagrange method $g(x,y) = x^2+y^2 = 16$

- $4x-4 = \lambda(2x)$
- $6y = \lambda(2y)$
- $x^2+y^2=16$

$y=0 \,/\, \lambda = 3$

$x = \pm 4 \,/\, x=-2 \;\&\; y = \pm\sqrt{12}$

Points: $(-2, \pm\sqrt{12})$ $(\pm 4, 0)$

$f(-4,0) = 43$ $f(4,0) = 11$ $f(-2, \pm\sqrt{12}) = 47$

$\boxed{\text{min is } -7 \quad \text{max is } 47}$

#22. Consider maximizing $f(x,y) = 2x+3y$ Subject to $\sqrt{x}+\sqrt{y} = 5$. Try using Lagrange multiplies then show $f(25,0)$ is a bigger value but doesn't satisfy $\nabla f = \lambda \nabla g$ for any λ. Explain why Lagranges Method fails.

- $2 = \lambda \frac{1}{2} x^{-1/2}$
- $3 = \lambda \frac{1}{2} y^{-1/2}$
- $\sqrt{x}+\sqrt{y} = 5$

$x = (\lambda/4)^2$ $y = (\lambda/6)^2$ sub $\frac{\lambda}{4} + \frac{\lambda}{6} = 5$ → $\lambda = 12$

$\lambda = 12$: $x = 9$ $y = 4$ $\boxed{f(9,4) = 30}$

$f(25,0) = 50 > f(9,4)$ but $2 = \lambda \frac{1}{2}(25)^{-1/2}$ → $\lambda = 20$

$3 = \lambda \frac{1}{2}(0)^{-1/2}$ → $3 \cancel{=} 0$ WAT?

$\sqrt{x}+\sqrt{y} = 5$ is bounded by points $(25,0)$ and $(0,25)$ there is no tangent line there to $g(x,y) = 5$.