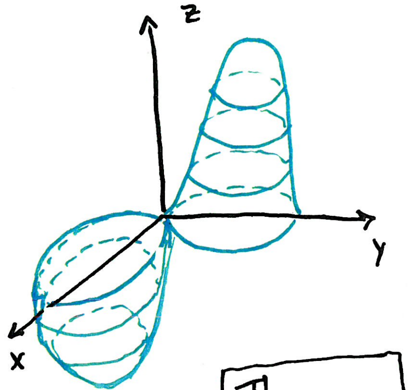
Section 14.7 - Max and Min Values



(a,b) a point in the domain of $f(x,y)$ is a:

- <u>local min</u> if $f(x,y) \geq f(a,b)$ when (x,y) is near (a,b)
- <u>Local max</u> if $f(x,y) \leq f(a,b)$ when (x,y) is near (a,b)
- <u>Absolute min</u> if $f(x,y) \geq f(a,b)$ for all (x,y)
- <u>Absolute max</u> if $f(x,y) \leq f(a,b)$ for all (x,y)

Theorem If f has a local max/min at (a,b) and $f_x(a,b)$ and $f_y(a,b)$ exist then $\nabla f(a,b) = \vec{0}$.

Proof: (a,b) local min/max of $f(x,y)$ is still a local min/max of $f(a,y)$ and $f(x,b)$ which are functions of one-variable. Thus

$$\frac{d}{dx} f(x,b)\Big|_{x=a} = f_x(a,b) = 0 \quad \text{also } f_y(a,b) = 0. \blacksquare$$

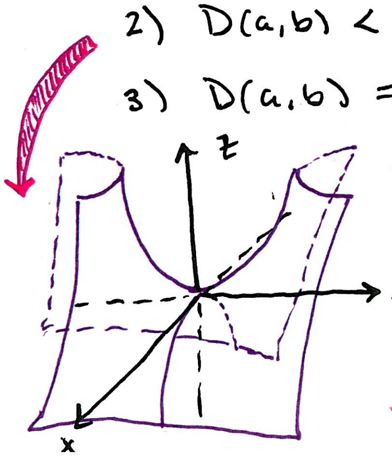
(a,b) is a <u>critical point</u> of f if $\nabla f(a,b) = \vec{0}$

2 point can be a local max, local min or <u>neither</u>.

- <u>Second Derivative Test</u>: 2^{nd} partials of f continuous on disk containing (a,b) where $\nabla f(a,b) = \vec{0}$. Define:

$$D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

1) $D(a,b) > 0$ then
   - Local max if $f_{xx}(a,b) < 0$ or $f_{yy}(a,b) < 0$
   - Local min if $f_{xx}(a,b) > 0$ or $f_{yy}(a,b) > 0$
2) $D(a,b) < 0$ then (a,b) is a <u>saddle point</u> (neither a max/min)
3) $D(a,b) = 0$ then nothing can be said

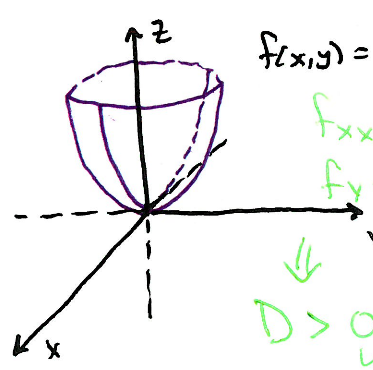


$f(x,y) = y^2 - x^2$

$f_{xx} < 0$

$f_{yy} > 0$

$\Downarrow$ $D < 0$ @ $(0,0)$ is a saddle point



$f(x,y) = x^2 + y^2$

$f_{xx} > 0$

$f_{yy} > 0$

$f_{xy} = 0$

$\Downarrow$

$D > 0$ @ $(0,0)$ local min

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MVC

Example Find the local max/min values and any saddle points of

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

① Critical numbers: $\vec{0} = \nabla f = \langle 4x^3 - 4y,\ 4y^3 - 4x \rangle$

$y = x^3 \quad \Rightarrow \quad 0 = 4(x^3)^3 - 4x = x(x-1)(x+1)(x^2+1)(x^4+1)$

Critical points: $(0,0)$, $(1,1)$, $(-1,-1)$

② Test critical points with: $D(x,y) = f_{xx} f_{yy} - f_{xy}^2$
$= (12x^2)(12y^2) - (-4)^2$

$D(0,0) = -16 \Rightarrow$ $\boxed{(0,0) \text{ is a saddle point}}$

$D(1,1) = 144 - 16 > 0$ and $f_{xx}(1,1) > 0 \Rightarrow$ $\boxed{(1,1) \text{ is a local min}}$

$D(-1,-1) = 144 - 16 > 0$ and $f_{xx}(-1,-1) > 0 \Rightarrow$ $\boxed{(-1,-1) \text{ is a local min}}$

Example Find the shortest distance from the point $(1,0,-2)$ to the plane: $x + 2y + z = 4$.

Minimize: $f(x,y) = d^2 = (x-1)^2 + (y)^2 + (z+2)^2 = (x-1)^2 + y^2 + (6-x-2y)^2$

① Critical numbers: $\vec{0} = \nabla f = \langle 2(x-1) + 2(6-x-2y)(-1),\ 2y + 2(6-x-2y)(-2) \rangle$

$0 = f_x \Rightarrow 0 = 4x + 4y - 14$ $\quad y = \frac{10}{6}$

$0 = f_y \Rightarrow 0 = 4x + 10y - 24$ $\quad x = 11/6$

② Test critical point: $D(x,y) = (4)(10) - (4)^2 > 0$ and $f_{xx} > 0$

Thus $(11/6,\ 10/6)$ is a minimimum (only 1 $\Rightarrow$ absolute min)

Shortest distance is $d = \left((11/6 - 1)^2 + (10/6)^2 + (6 - 11/6 - 10/3)^2 \right)^{1/2} = \boxed{\frac{5}{6}\sqrt{6}}$

Extreme Value Theorem (EVT): Existence Theorem!

(1) $y = f(x)$ continous on a closed interval $[a,b]$ will have an absolute max and an absolute min on $[a,b]$.

(2) $z = f(x,y)$ continuous on a closed bounded set D of $\mathbb{R}^2$, will have an absolute max and an absolute min on D.

Translate closed interval to closed set in $\mathbb{R}^2$: must contain all boundary pts

Closed:

Not Closed:

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- **Critical Point Theorem:** for functions on a closed bounded set

The absolute max/min value of:

(1) $y=f(x)$ occurs at either a critical point or End point

(2) $z=f(x,y)$ occurs at either a critical point or boundary point

Example Find the absolute max/min value of $f(x,y)=x^2-2xy+2y$ on the rectangle $D=\{(x,y) \mid 0 \le x \le 3, 0 \le y \le 2\}$.

① Critical points: $\vec{0}=\nabla f=\langle 2x-2y, -2x+2\rangle$ at $(1,1)$ $f(1,1)=1$

make sure its in D

② Extreme Values on Boundaries:

I. $x=0 \Rightarrow f(0,y)=2y$ on $[0,2]$ max: 4 min: 0

II. $x=3 \Rightarrow f(3,y)=9-4y$ on $[0,2]$ max: 9 min: 1

III. $y=0 \Rightarrow f(x,0)=x^2$ on $[0,3]$ max: 9 min: 0

IV. $y=2 \Rightarrow f(x,2)=x^2-4x+4=(x-2)^2$ on $[0,3]$ max: 4 min: 0

③ Compare values:

Absolute max of f is $\boxed{9}$

Absolute min of f is $\boxed{0}$

Example Same function on the triangle whose vertices are $(0,0), (1,0), (0,1)$

① Critical Points: Same above $(1,1)$ still in triangle $f(1,1)=1$

② Extreme Values on Boundaries:

I. $x=0 \Rightarrow f(0,y)=2y$ on $[0,1]$ max: 2 min: 0

II. $y=1 \Rightarrow f(x,1)=x^2-2x+2=(x-1)^2+1$ on $[0,1]$ max: 2 min: 1

III. $y=x \Rightarrow f(x,x)=-x^2+2x=x(-x+2)$ on $[0,1]$ max: 1 min: 0

③ Compare Values:

Absolute max of f is $\boxed{2}$

Absolute min of f is $\boxed{0}$