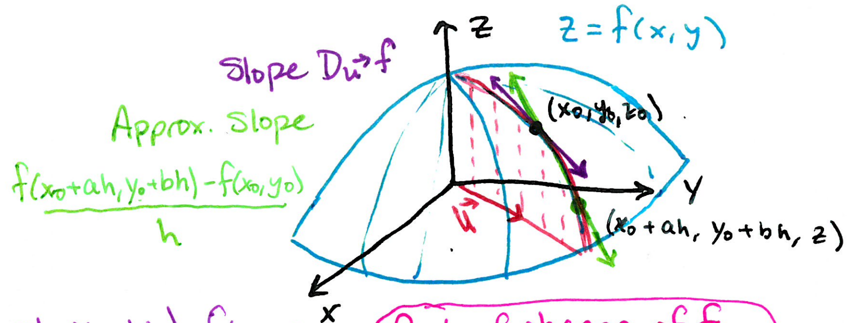
Section 14.6 - Directional Derivatives & the Gradient

MVC

- **Directional Derivative:**

at (x_0, y_0) in the direction of $\vec{u} = \langle a, b\rangle$ (unit vector) of $f(x,y)$ is



$$D_{\vec u} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Rate of change of f in the direction of $\vec u$

Theorem If f is a differentiable function of x and y, then f has a directional derivative for any direction $\vec u = \langle a, b\rangle$ (unit vector) and

$$\boxed{D_{\vec u} f(x,y) = a\, f_x(x,y) + b\, f_y(x,y)}$$

Proof: Let $g(h) = f(x+ah, y+bh)$ then $\underline{g'(0) = D_{\vec u} f(x,y)}$

also $g'(h) = f_x(x+ah, y+bh)a + f_y(x+ah, y+bh)\cdot b$ so $\underline{g'(0) = f_x(x,y)a + f_y(x,y)\cdot b}$ ■

Example 2 Find the directional derivative if $f(x,y) = x^3 - 3xy + 4y^2$ and $\vec u$ is the unit vector given by $\theta = \pi/6$. Find $D_{\vec u} f(1,2)$.

$\vec u = \langle \cos \pi/6, \sin \pi/6\rangle = \langle \sqrt3/2, 1/2\rangle$ $\quad f_x(x,y) = 3x^2 - 3y \quad f_y(x,y) = -3x + 8y$

$D_{\vec u} f(1,2) = f_x(1,2)\cdot \frac{\sqrt3}{2} + f_y(1,2)\cdot \frac12 = \boxed{(-3)\frac{\sqrt3}{2} + \frac{13}{2}}$

Note: $D_{\vec u} f(x,y) = \langle a, b\rangle \cdot \langle f_x, f_y\rangle = \vec u \cdot \langle f_x, f_y\rangle$

- **Gradient of f:** is the vector $\nabla f = \langle f_x, f_y\rangle$ for $z = f(x,y)$

In general for $z = f(\vec x)$ where $\vec x = \langle x_1, \dots, x_n\rangle$, $\nabla f = \langle f_{x_1}, \dots, f_{x_n}\rangle$

$$D_{\vec u} f(\vec x) = \nabla f \cdot \vec u$$

Example $f(x,y,z) = y\ln(x^2 + z)$ find ∇f and $D_{\vec u} f$ in the direction of $\vec v = \langle 1, -1, 1\rangle$ at $(0,5,1)$

$\nabla f = \langle f_x, f_y, f_z\rangle = \left\langle \frac{2xy}{x^2+z}, \ln(x^2+z), \frac{y}{x^2+z}\right\rangle$

$\vec u = \frac{1}{\sqrt3}\langle 1, -1, 1\rangle \quad \nabla f(0,5,1) = \langle 0, 0, 5\rangle$

$D_{\vec u} f(0,5,1) = \vec u \cdot \nabla f(0,5,1) = \boxed{5/\sqrt3}$

• Question: How would you maximize the directional derivative?
(that is find the max of $D_{\vec{u}}f$ for a point on f)

$$D_{\vec{u}}f = \vec{u}\cdot\nabla f \underset{\substack{\uparrow \\ \text{Dot Product formula}}}{=} |\vec{u}||\nabla f|\cos\theta \underset{\substack{\uparrow \\ \text{since } |\vec{u}|=1}}{=} |\nabla f|\cos\theta \rightarrow \text{maximized when } \theta = 0$$

Theorem If f is a differentiable function then the max value of the directional derivative $D_{\vec{u}}f(\vec{x})$ is: $|\nabla f(\vec{x})|$
and it occurs in the direction of: $\nabla f(\vec{x})$

Proof: See above work ↑ ∎

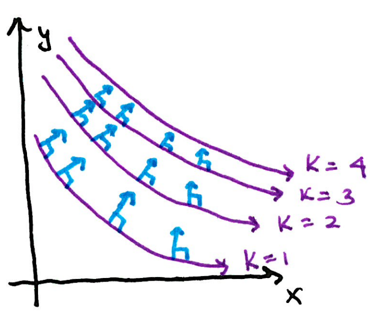
Example 7 Suppose that the temp at a point (x,y,z) in space is given by $T(x,y,z) = 80(1+x^2+2y^2+3z^2)^{-1}$ °C where x,y,z are in meters. In what direction is the temp increasing fastest at $(1,1,-2)$ and what is the max rate of increase?

$$\nabla T = \frac{-80}{(1+x^2+2y^2+3z^2)^2}\langle 2x, 4y, 6z\rangle$$

$$\nabla T(1,1,-2) = \frac{-80}{16^2}\langle 2, 4, -12\rangle = \boxed{\left\langle -\tfrac{5}{8}, -\tfrac{5}{4}, \tfrac{15}{4}\right\rangle} \text{ Direction of max rate}$$

$$|\nabla T(1,1,-2)| = \frac{5}{8}\sqrt{41}\ \text{°C/m} \approx \boxed{4\ \text{°C/m}} \text{ max rate of increase}$$

• Level Curves: $f(x,y) = K$ for $z = f(x,y)$

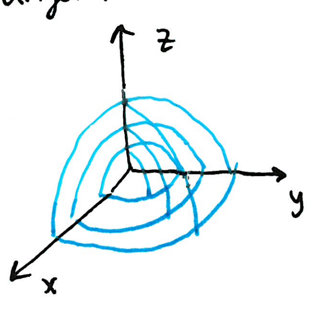


• Draw the gradient vectors on the level curves

★ Do you see a relation between the gradient and another vector?

$\nabla f \perp$ to level curves, tangent vectors are tangent to curves
thus $\nabla f \perp$ tangent vectors of curves

• Tangent Plane to a Level Surface: $f(x,y,z) = K$ for $z = f(x,y,z)$



$C: \vec{r}(t)$ some curve on the surface through (x,y,z)

$$\nabla f\cdot\vec{r}'(t) = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} + f_z\frac{dz}{dt} = \frac{d}{dt}(f(x,y,z)) = \frac{d}{dt}(K) = 0 \text{ so } \nabla f \perp \vec{r}'(t)$$

Tangent Plane to $f(x,y,z) = K$ at (x_0,y_0,z_0):

$$\boxed{f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0}$$

★Watch: youtube.com/watch?v=NuNCIRnXWcE

- Review:

① $D_{\vec{u}} f(x,y) = \vec{u} \cdot \nabla f$ $\quad \vec{u}$ must be a unit vector

② $\nabla f = \langle f_x, f_y \rangle$

③ Max value of $D_{\vec{u}} f(x,y)$ is $|\nabla f|$

④ Direction of max value of $D_{\vec{u}} f(x,y)$ is ∇f

⑤ ∇f $\perp$ tangent vectors on level curves (surfaces)

⑥ ∇f is the normal vector for the tangent plane to $f(x,y,z) = K$

⑦ ∇f is the direction of the normal line

⑧ On a level curve graph, ∇f points in the direction of greatest z increase

⑨ ∇f makes a $90°$ angle with the level curves

Example 8 Find the equations of the tangent plane and normal line at $(-2,1,-3)$ to $x^2/4 + y^2 + z^2/9 = 3$.

$f(x,y,z) = \frac{x^2}{4} + y^2 + z^2/9 \qquad \nabla f = \langle x/2,\ 2y,\ 2z/9 \rangle \qquad \nabla f(-2,1,-3) = \langle -1, 2, -2/3 \rangle$

Plane: $\boxed{-1(x+2) + 2(y-1) - 2/3\,(z+3) = 0}$

Normal line: $\boxed{x = -2 - t,\quad y = 1 + 2t,\quad z = -3 - 2/3\, t}$

#39 Second Directional Derivative:

$$D_{\vec{u}}^2 f(x,y) = D_{\vec{u}}\left(D_{\vec{u}} f(x,y)\right)$$

Find $D_{\vec{u}}^2 f(x,y)$ if $f(x,y) = x^3 + 5x^2y + y^3$ and $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$D_{\vec{u}} f = \vec{u} \cdot \nabla f = \vec{u} \cdot \langle 3x^2 + 10xy,\ 5x^2 + y^3 \rangle = \frac{3}{5}(3x^2 + 10xy) + \frac{4}{5}(5x^2 + y^3)$

$D_{\vec{u}}^2 f = \vec{u} \cdot \nabla\left(\frac{3}{5}(3x^2 + 10xy) + \frac{4}{5}(5x^2 + y^3)\right)$

$= \boxed{\left(\frac{3}{5}\right)^2(6x + 10y) + \left(\frac{3}{5}\right)\left(\frac{4}{5}\right)(10x) + \left(\frac{3}{5}\right)\left(\frac{4}{5}\right)(10x) + \left(\frac{4}{5}\right)^2(3y^2)}$

Section 14.6 - Directional Derivatives & The Gradient

MVC

- Extra Examples:

#40 (a) If $\vec{u} = \langle a,b \rangle$ is a unit vector and f has continuous 2nd partials
Show that $D_{\vec{u}}^2 f = f_{xx} a^2 + 2 f_{xy} ab + f_{yy} b^2$

$D_{\vec{u}} f = a f_x + b f_y$ $\qquad$ $D_{\vec{u}}^2 f = a^2 f_{xx} + ab f_{xy} + b^2 f_{yy}$

$$D_{\vec{u}}^2 f = D_{\vec{u}}(D_{\vec{u}} f) = D_{\vec{u}}(af_x + bf_y)$$
$$= \vec{u} \cdot \nabla(af_x + bf_y)$$
$$= \vec{u} \cdot \langle af_{xx} + bf_{xy},\ af_{xy} + bf_{yy} \rangle$$

#55 Are there any points on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to $x + y = z$?

Point where: $\langle 1,1,-1 \rangle = \lambda \nabla f$ where $f(x,y,z) = x^2 - y^2 - z^2$

$$\langle 1,1,-1 \rangle = \lambda \langle 2x, -2y, -2z \rangle$$

Point possible: $\lambda(1,-1,1)$

On hyperboloid? $(\lambda)^2 - (-\lambda)^2 - (\lambda)^2 = -\lambda^2 \neq 1$ Thus $\boxed{\text{No point}}$

#61 Show that the sum of the x, y, z intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{d}$ is a constant.

Tangent Plane at (a,b,c) where $\sqrt{a} + \sqrt{b} + \sqrt{c} = \sqrt{d}$

$$a^{-1/2}(x-a) + b^{-1/2}(y-b) + c^{-1/2}(z-c) = 0$$
$$a^{-1/2}x + b^{-1/2}y + c^{-1/2}z = \sqrt{a} + \sqrt{b} + \sqrt{c} = \sqrt{d}$$

x-intercept: $\sqrt{d}\sqrt{a}$

y-intercept: $\sqrt{d}\sqrt{b}$

z-intercept: $\sqrt{d}\sqrt{c}$

Sum of intercepts $= \sqrt{d}\sqrt{a} + \sqrt{d}\sqrt{b} + \sqrt{d}\sqrt{c}$
$= \sqrt{d}(\sqrt{d}) = d$ ← constant ✓

#67 Suppose $D_{\vec{u}} f(x,y)$ and $D_{\vec{v}} f(x,y)$ are know for two non-parallel vectors $\vec{u}, \vec{v}$. Is it possible to find $\nabla f(x,y)$? If so how?

$\vec{u} = \langle a,b \rangle$ $\quad \vec{v} = \langle c,d \rangle$ unit vectors non-parallel $\Rightarrow \vec{u} \times \vec{v} = ad - bc \neq 0$

$$\left.\begin{aligned} D_{\vec{u}} f &= af_x + bf_y \\ D_{\vec{v}} f &= cf_x + df_y \end{aligned}\right\} \quad \boxed{\begin{aligned} f_y &= \frac{cD_{\vec{u}}f - aD_{\vec{v}}f}{bc - ad} \\ f_x &= \frac{dD_{\vec{u}}f - bD_{\vec{v}}f}{ad - bc} \end{aligned}}$$

both defined since $D_{\vec{u}}f, D_{\vec{v}}f$ exist and $ad - bc \neq 0$