Section 14.5 - Chain Rule

Recall: For $y=f(x)$ and $x=g(t)$, $y=f(g(t))$, where both f and g are differentiable then:

$$\frac{dy}{dt}=\frac{d}{dt}\left(f(g(t))\right)=f'(g(t))\cdot g'(t)=\frac{df}{dx}\cdot\frac{dx}{dt}$$

- Chain Rule (Case 1): $z=f(x,y)$ differentiable with $x=g(t)$ and $y=h(t)$ both differentiable then:

$$\frac{\Delta z}{\Delta t}=f_x\frac{\Delta x}{\Delta t}+f_y\frac{\Delta y}{\Delta t}+\varepsilon_1\frac{\Delta x}{\Delta t}+\varepsilon_2\frac{\Delta y}{\Delta t} \qquad \frac{dz}{dt}=\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{\Delta z}{\Delta t}$$

$$\boxed{\frac{dz}{dt}=f_x\frac{dx}{dt}+f_y\frac{dy}{dt}}$$

- Chain Rule (Case 2): $z=f(x,y)$ differentiable with $x=g(s,t)$ and $y=h(s,t)$ both differentiable then:

Extends to more variables

$$\boxed{\frac{\partial z}{\partial s}=f_x\frac{\partial x}{\partial s}+f_y\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t}=f_x\frac{\partial x}{\partial t}+f_y\frac{\partial y}{\partial t}}$$

Example 2 The pressure P (in kPa), volume V (in L), temp T (in K) of a mole of an ideal gas are related by $PV=8.31T$. Find the rate at which the pressure is changing when the temp is 300 K increasing at 0.1 K/sec and the volume is 100 L increasing at 0.2 L/sec.

$$P=8.31\frac{T}{V}$$

$$\frac{dP}{dt}=\frac{\partial}{\partial V}(8.31TV^{-1})\frac{dV}{dt}+\frac{\partial}{\partial T}(8.31TV^{-1})\frac{dT}{dt}$$

$$=-8.31TV^{-2}\frac{dV}{dt}+8.31V^{-1}\frac{dT}{dt}$$

$$\left.\frac{dP}{dt}\right|_{\substack{V=100\\ K=300}}=-8.31(300)(100)^{-2}(0.2)+8.31(100)^{-1}(0.1)$$

$$\approx -0.04155 \text{ kPa/s}$$

The pressure is decreasing at a rate of 0.04155 kPa/s when the temp is 300 K and volume is 100 L.

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Example 5 If $u = x^4 y + y^2 z^3$ where $x = rse^t$, $y = rs^2 e^{-t}$ and $z = r^2 s \sin t$ then find $\partial u/\partial s$ when $r=2$, $s=1$, $t=0$. $x=2$ $y=2$ $z=0$

$$\left.\frac{\partial u}{\partial s}\right|_{(2,1,0)} = \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}\right)\Bigg|_{(2,1,0)} = 64(2) + (16)(4) = \boxed{192}$$

$$\left.\frac{\partial x}{\partial s}\right|_{(2,1,0)} = re^t\Big|_{(2,1,0)} = 2 \qquad \left.\frac{\partial z}{\partial s}\right|_{(2,1,0)} = r^2\sin(t)\Big|_{(2,1,0)} = 0 \qquad \left.\frac{\partial u}{\partial y}\right|_{(2,2,0)} = x^4 + 2yz^3\Big|_{(2,2,0)} = 16$$

$$\left.\frac{\partial y}{\partial s}\right|_{(2,1,0)} = 2rse^{-t}\Big|_{(2,1,0)} = 4 \qquad \left.\frac{\partial u}{\partial x}\right|_{(2,2,0)} = 4x^3y\Big|_{(2,2,0)} = 64$$

- **Implicit Differentiation:** suppose $F(x,y)=0$ defines y implicitly as a differentiable function of x, $f(x)=y$ with $F(x, f(x)) = 0$ then:

$$F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = 0$$

Implicit Function Theorem $F(x, y) = 0$, F differentiable, $F_y \neq 0$ then $\boxed{dy/dx = -F_x/F_y}$

If $F(x,y,z) = 0$, $z = f(x,y)$ implicitly defined, F differentiable, $F_z \neq 0$

then: $\boxed{\partial z/\partial x = -F_x/F_z}$ $\qquad$ $\boxed{\partial z/\partial y = -F_y/F_z}$

Since $F_x \frac{dx}{dx} + F_y \frac{dy}{dx} + F_z \frac{dz}{dx} = 0$ where $\frac{dy}{dx} = 0$

Example 8 Find dy/dx if $x^3 + y^3 = 6xy$.

$$\frac{dy}{dx} = \frac{-F_x}{F_y} \quad \text{where} \quad F(x,y) = x^3 + y^3 - 6xy = 0$$

$$= \boxed{\frac{-(3x^2 - 6y)}{3y^2 - 6x}}$$

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- Extra Examples

\# 33 Find $\partial z/\partial x$ and $\partial z/\partial y$ where $e^z = xyz$.

$$e^z \frac{\partial z}{\partial x} = yz + xy\frac{\partial z}{\partial x} \Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}}$$

$$e^z \frac{\partial z}{\partial y} = xz + xy\frac{\partial z}{\partial y} \Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}}$$

\# 39 The length l, width w, and height h of a box change with time. When $l = 1m$, $w = h = 2m$ and l and w are increasing at $2m/s$ while h is decreasing at $3m/s$. Find the rate of change in:

(a) volume $V = lwh$

$$\frac{dV}{dt} = wh\frac{dl}{dt} + lw\frac{dh}{dt} + lh\frac{dw}{dt} \qquad \left.\frac{dV}{dt}\right|_{(1,2,2)} = 4(2) + 2(-3) + 2(2) = \boxed{6\,m^3/s}$$

(b) surface area $S = 2lw + 2wh + 2lh$

$$\frac{dS}{dt} = 2w\frac{dl}{dt} + 2l\frac{dw}{dt} + 2w\frac{dh}{dt} + 2h\frac{dw}{dt} + 2l\frac{dh}{dt} + 2h\frac{dl}{dt} \qquad \left.\frac{dS}{dt}\right|_{(1,2,2)} = 4(2) + 2(2) + 4(-3) + 4(2) + 2(-3) + 4(2)$$

$$= \boxed{10\,m^2/s}$$

(c) length of diagonal

$$D = \sqrt{l^2 + w^2 + h^2}$$

$$\frac{dD}{dt} = \frac{1}{2}\left(l^2 + w^2 + h^2\right)^{-1/2} \cdot \left(2l\frac{dl}{dt} + 2w\frac{dw}{dt} + 2h\frac{dh}{dt}\right) \qquad \left.\frac{dD}{dt}\right|_{(1,2,2)} = \frac{1}{2}(9)^{-1/2}\left(2(2) + 4(2) + 4(-3)\right)$$

$$= \frac{1}{6}(0) = \boxed{0\,m/s}$$

\# 45 If $z = f(x,y)$ and $x = r\cos\theta$, $y = r\sin\theta$ show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$\frac{\partial z}{\partial r} = f_x\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta$$

$$\frac{\partial z}{\partial \theta} = f_x\frac{\partial x}{\partial \theta} + f_x\frac{\partial y}{\partial \theta} = f_x(-r\sin\theta) + f_y(r\cos\theta)$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = f_x^2\cos^2\theta + f_y^2\sin^2\theta + \cancel{2f_xf_y\cos\theta\sin\theta}$$

$$+ f_x^2\sin^2\theta + f_y^2\cos^2\theta - \cancel{2f_xf_y\cos\theta\sin\theta}$$

$$= f_x^2 + f_y^2$$