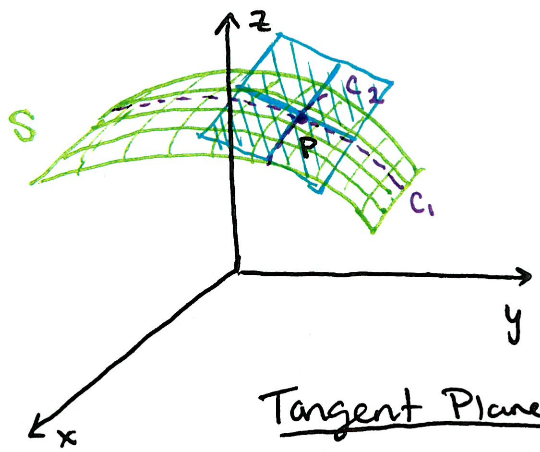

- S given by $z = f(x,y)$ with continuous first partials
- P a point (x_0, y_0, z_0) on S
- $C_1 = f(x_0, y)$ and $C_2 = f(x, y_0)$

Tangent Plane to S at P: the plane containing the tangent lines to C_1 and C_2 at the point P.

Slope of tangent line to $C_1 = f_y(x_0, y_0)$ Direction $= \langle 0, 1, f_y \rangle$

Slope of tangent line to $C_2 = f_x(x_0, y_0)$ Direction $= \langle 1, 0, f_x \rangle$

normal vector to plane $= \langle 0, 1, f_y \rangle \times \langle 1, 0, f_x \rangle = \langle f_x, f_y, -1 \rangle$

Tangent plane to $z = f(x,y)$ at (x_0, y_0): $\boxed{f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) - (z - z_0) = 0}$

Example 1 Find the tangent plane to $z = 2x^2 + y^2$ at $(1,1,3)$.

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1)} = 4x\Big|_{(1,1)} = 4$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,1)} = 2y\Big|_{(1,1)} = 2$$

tangent plane:

$$\boxed{4(x-1) + 2(y-1) - (z-3) = 0}$$

- **Linear Approximation:** $f(x,y) \approx L(x,y) = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) + z_0$ for points (x,y) near (x_0, y_0)

  Points near (x_0, y_0) on $f \approx$ tangent plane value

- **$f(x,y)$ Differentiable:** at (a,b) if $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to 0$.

Theorem If f_x and f_y exist near (a,b) and are continuous at (a,b) then f is differentiable at (a,b).

Proof: $\Delta z = f(a + \Delta x, b + \Delta y) - f(a,b)$ let $a' = a + \Delta x$ and $b' = b + \Delta y$

$$= \underbrace{\left[f(a', b') - f(a, b')\right]}_{\text{function of } x} + \underbrace{\left[f(a, b') - f(a,b)\right]}_{\text{function of } y}$$

by MVT $= f_x(x_a, b')\Delta x + f_y(a, y_b)\Delta y$ where $x_a \in (a, a+\Delta x)$ $y_b \in (b, b+\Delta y)$

$$= f_x(a,b)\Delta x + \underbrace{\left[f_x(x_a, b') - f_x(a,b)\right]}_{\text{continuous} \to (a,b)}\Delta x + f_y(a,b)\Delta y + \underbrace{\left[f_y(a, y_b) - f_y(a,b)\right]}_{\text{continuous} \to (a,b)}\Delta y$$

$= f_x(a,b)\Delta x + \varepsilon_1 \Delta x + f_y(a,b)\Delta y + \varepsilon_2 \Delta y$ where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$ ∎

Section 14.4 – Tangent Planes

MVC

Example 2 Show $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$ and find its Linearization at $(1,0)$ to approximate $f(1.1,-0.1)$.

$f_x(x,y) = e^{xy} + xye^{xy}$

$f_y(x,y) = x^2e^{xy}$

Both continuous so f is differentiable

$f_x(1,0) = 1 \quad f_y(1,0) = 1$

$L(x,y) = (x-1) + y + 1 \quad f(1.1,-0.1) \approx L(1.1,-0.1) = \boxed{1}$

• Differentials:

One variable

$y = f(x)$

$\Delta y = f'(x)\Delta x + \varepsilon_1 \Delta x$

$dy = f'(x)dx$

$\Delta y \approx dy$

Two variables

$z = f(x,y)$

$\Delta z = f_x(x,y)\Delta x + f_y(x,y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2 \Delta y$

$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$

$\Delta z \approx dz$

Example 5 The base radius and height of a right circular cone are measured as 10cm and 25cm, with a possible error of 0.1cm in each. Use differentials to estimate the max error in calculating the volume of the cone, then check by computing two volumes.

$V = \frac{1}{3}\pi r^2 h \qquad |\Delta r| = |dr| \leq 0.1 \qquad |\Delta h| = |dh| \leq 0.1$

$dV = \frac{2}{3}\pi rh\,dr + \frac{1}{3}\pi r^2 dh$

Max error in Volume $\approx dV = \frac{2}{3}\pi(10)(25)(0.1) + \frac{1}{3}\pi(10)^2(0.1)$

$= 20\pi \text{ cm}^3 \approx \boxed{62.83 \text{ cm}^3}$

Check: $V_1 = \frac{1}{3}\pi(10)^2(25) = \frac{2500}{3}\pi$

$V_2 = \frac{1}{3}\pi(10.1)^2(25.1) = \frac{2560.451}{3}\pi$

Difference $= \frac{2560.451}{3}\pi - \frac{2500}{3}\pi \approx \boxed{63.30 \text{ cm}^3}$

Section 14.4 - Tangent Planes

- Extra Examples

#31 If $z = 5x^2 + y^2$ and (x, y) changes from $(1,2)$ to $(1.05, 2.1)$ Compare Δz and dz.

$$\Delta z = z(1.05, 2.1) - z(1,2) = 5(1.05)^2 + (2.1)^2 - 5 - 4 = \boxed{0.612}$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad dz = 10x|_{(1,2)}(0.05) + 2y|_{(1,2)}(0.1)$$

$$= 10(0.05) + 4(0.1)$$

$$= \boxed{0.9}$$

#38 The pressure, volume, and temp of a mole of an ideal gas are related by the equation $PV = 8.31T$ where P is measured in KPa, V in L, T in K. Use differentials to find the approx. change in pressure if V is increased from 12L to 12.3L and the temp decreases from 310 K to 305 K.

$$P = \frac{8.31T}{V}$$

$$dP = -8.31V^{-2}T\,dV + \frac{8.31}{V}dT$$

$$dP = -8.31(12)^{-2}(310)(0.3) + \frac{8.31}{(12)}(-5) \approx \boxed{-8.829 \text{ KPa}}$$

#42 $\vec{r}_1(t) = \langle 2+3t, 1-t^2, 3-4t+t^2 \rangle$ and $\vec{r}_2(u) = \langle 1+u^2, 2u^3-1, 2u+1 \rangle$ lie on S and contain $(2,1,3)$. Find the tangent plane to S at $(2,1,3)$.

$$\vec{r}_1{}'(t) = \langle 3, -2t, -4+2t \rangle \qquad \vec{r}_1{}'(0) = \langle 3, 0, -4 \rangle$$

$$\vec{r}_2{}'(u) = \langle 2u, 6u^2, 2 \rangle \qquad \vec{r}_2{}'(1) = \langle 2, 6, 2 \rangle$$

Point $(2,1,3)$

$t = 0$

$u = 1$

Normal $= \vec{r}_1{}'(0) \times \vec{r}_2{}'(1) = \langle 10, -14, 18 \rangle$

tangent plane:

$$\boxed{10(x-2) - 14(y-1) + 18(z-3) = 0}$$