Section 14.3 - Partial Derivatives

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- Recall: Definition of the derivative of $y=f(x)$ at $x=a$

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$ provided the limit exists

- Now: For a function $z=f(x,y)$ only vary x, fix y as a constant $y=b$

Then $g(x) = f(x,b)$ is a function of 1 variable

and if $g'(a)$ exists then we call it the partial derivative of f with respect to x at (a,b):

$$f_x(a,b) = g'(a) = \lim_{x \to a} \frac{f(x,b)-f(a,b)}{x-a}$$

Similarly, partial derivative of f wrt y at (a,b):

$$f_y(a,b) = \lim_{y \to b} \frac{f(a,y)-f(a,b)}{y-b}$$

- Notation for Partial Derivatives: $z=f(x,y)$

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(f(x,y)\right) = \frac{\partial z}{\partial x} = z_x$$

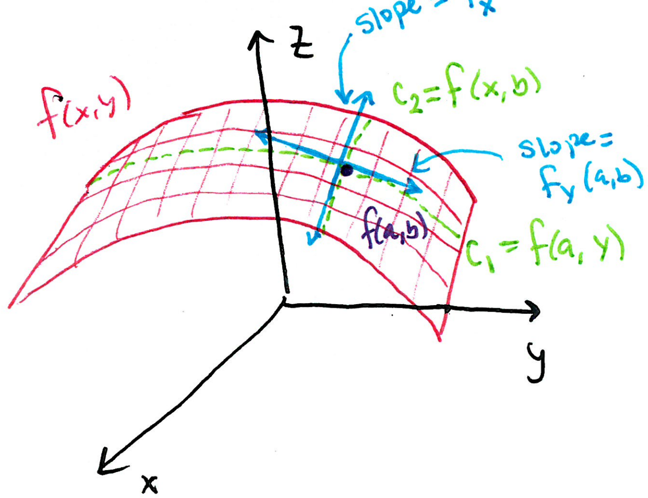
$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(f(x,y)\right) = \frac{\partial z}{\partial y} = z_y$$

Example $f(x,y) = x^2\sin(y) + x\ln(x+y^2)$, Find $f_x(2,0)$ and $f_y(2,0)$

$$f_x(x,y) = 2x\sin(y) + \ln(x+y^2) + \frac{x}{x+y^2} \qquad f_x(2,0) = \boxed{\ln(2)+1}$$

$$f_y(x,y) = x^2\cos(y) + \frac{x}{x+y^2}\cdot 2y \qquad f_y(2,0) = \boxed{4}$$

- Interpretation:



Example 4 $x^3+y^3+z^3+6xyz=1$
Find $\partial z/\partial x$ and $\partial z/\partial y$.

Differentiate wrt x:

$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-3x^2-6yz}{3z^2+6xy}}$$

Differentiate wrt y:

$$0 + 3y^2 + 3z^2\cdot\frac{\partial z}{\partial y} + 6xz + 6xy\frac{\partial z}{\partial y} = 0$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-3y^2-6xz}{3z^2+6xy}}$$

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- **Higher Order Derivatives:** Second order partials, ..., n^{th} order partials

<u>Notation</u>: $(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$

$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ $\quad (f_y)_y = f_{yy} = \frac{\partial^2 f}{\partial y^2}$

$(f_y)_x = f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$

Example 6 Find the Second Partial derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x = 3x^2 + 2xy^3$ $\quad f_{xx} = 6x + 2y^3$ $\quad f_{xy} = 6xy^2$

$f_y = 3x^2y^2 - 4y$ $\quad f_{yy} = 6x^2y - 4$ $\quad f_{yx} = 6xy^2$

Clairaut's Theorem f defined on D containing (a,b). If f_{xy} and f_{yx} are continuous on D then: $\boxed{f_{xy} = f_{yx}}$

Example Show $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ fails Clairaut's Theorem at $(0,0)$. Why?

$f_x = \frac{(y(x^2-y^2) + 2x^2y)(x^2+y^2) - 2x^2y(x^2-y^2)}{(x^2+y^2)^2}$ $\quad f_x(0,y) = \frac{(-y^3)(y^2) - 0}{y^4} = -y$ $\quad f_{xy}(0,0) = -1$

$f_y = \frac{(x(x^2-y^2) + 2xy^2)(x^2+y^2) - 2xy^2(x^2-y^2)}{(x^2+y^2)^2}$ $\quad f_y(x,0) = \frac{x^3x^2 - 0}{x^4} = x$ $\quad f_{yx}(0,0) = 1$

$\neq$

- **Partial Differential Equations:** an equation with partial derivatives

<u>Example</u>: Laplace Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Solutions are called <u>Harmonic Functions</u>

↳ used in Heat Conduction, fluid Flow, electric Potential

Example 8 Show $f(x,y) = e^x \sin y$ is a solution of the Laplace Equation.

$u = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \sin y$ $\quad \frac{\partial u}{\partial y} = e^x \cos y$

$\frac{\partial^2 u}{\partial x^2} = e^x \sin y$ $\quad \frac{\partial^2 u}{\partial y^2} = -e^x \sin y$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \sin y - e^x \sin y = 0$ ✓

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- Extra Examples:

#9 See page 936 label graphs a, b, c as f, f_x, f_y give reasons.

Follow sign of slopes along curves in the x or y direction

f is the graph of c since $2 \geq y \geq 0$ in direction of x, f_x is −, 0, +
$-3 \leq y \leq 0$ in direction of x, f_x is +, 0, − $\Rightarrow$ **f_x is graph b**

and since for $x \neq 0$ in direction of y, f_y is −, 0, +, 0, − $\Rightarrow$ **f_y is graph c**

#71 $f(x,y,z) = xy^2z^3 + \arcsin(x\sqrt{z})$ find f_{xzy} (Hint: Which order is easier?)

$f_y = 2xyz^3$

$f_{yx} = 2yz^3$

$\boxed{f_{yxz} = 6yz^2}$

$f_{xzy} = \frac{\partial}{\partial y}\frac{\partial}{\partial z}\left(y^2z^3 + \frac{\partial}{\partial x}(\arcsin(x\sqrt{z}))\right)$

$= \frac{\partial}{\partial y}\left(3y^2z^2 + \frac{\partial}{\partial z}\frac{\partial}{\partial x}(\arcsin(x\sqrt{z}))\right)$

$= \boxed{6yz^2} + 0$

#83 Total resistance R produced by 3 conductors with resistance R_1, R_2, R_3 and connected in a parallel electrical circuit is $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$
Find $\partial R / \partial R_1$

$R^{-1} = R_1^{-1} + R_2^{-1} + R_3^{-1}$

$\frac{\partial}{\partial R_1}(R^{-1}) = \frac{\partial}{\partial R_1}(R_1^{-1} + R_2^{-1} + R_3^{-1})$

$-R^{-2}\frac{\partial R}{\partial R_1} = -R_1^{-2} \Rightarrow \boxed{\frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}}$

#88 The gas Law for a fixed mass m of an ideal gas at absolute temp T, pressure P, and volume V is $PV = mRT$ where R is the gas constant. Show

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$$

$\frac{\partial P}{\partial V} = -mRTV^{-2}$, $\frac{\partial V}{\partial T} = \frac{mR}{P}$, $\frac{\partial T}{\partial P} = \frac{V}{mR}$

$\frac{\partial P}{\partial V}\cdot\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial P} =$

$(-mRTV^{-2})\left(\frac{mR}{P}\right)\left(\frac{V}{mR}\right)$

$= (-PV\,V^{-2})\left(\frac{V}{P}\right)$

$= -1$ ∎

#93 Is there a function f with $f_x(x,y) = x + 4y$ and $f_y(x,y) = 3x - y$?

$f = \int f_x(x,y)\,dx = \int x + 4y\,dx = \frac{x^2}{2} + 4xy + C(y)$

$f = \int f_y(x,y)\,dy = \int 3x - y\,dy = 3xy - \frac{y^2}{2} + C(x)$

There is no f