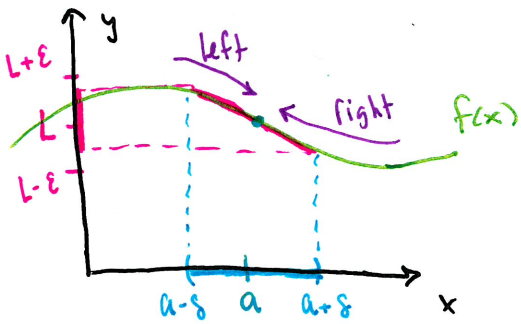
Section 14.2 - Limits & Continuity

MVC

2D Limits:

$$\lim_{x \to a} f(x) = L$$



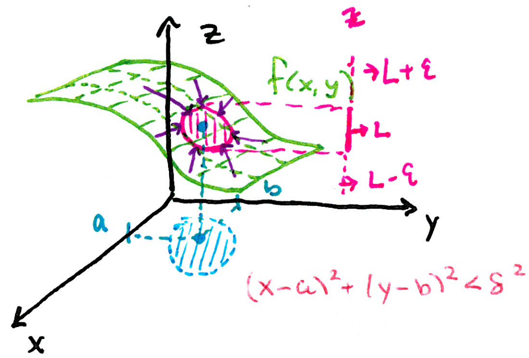
To Exist: $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

• <u>Definition</u>: $\lim_{x \to a} f(x) = L$ if

for all $\varepsilon > 0$ there is $\delta > 0$ so if $x \in D$ with $|x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

3D Limits:

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$



To Exist: Need all paths on $f(x,y)$ as $(x,y) \to (a,b)$ to approach the same value!

• <u>Definition</u> $\lim_{(x,y) \to (a,b)} f(x,y) = L$ if

for all $\varepsilon > 0$ there is $\delta > 0$ so if $(x,y) \in D$ with $(x-a)^2 + (y-b)^2 < \delta^2$ then $|f(x,y) - L| < \varepsilon$.

★ Easy to show limit DNE, if $f(x,y) \to L_1$ on path C_1 as $(x,y) \to (a,b)$ and $f(x,y) \to L_2 \neq L_1$ on path C_2 as $(x,y) \to (a,b)$ then $\lim_{(x,y) \to (a,b)} f(x,y)$ DNE

$\boxed{\text{Example 1}}$ Show that $\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ DNE.

$$\left.\begin{array}{l} C_1: (x,y) \to (0,0) \text{ along } x\text{-axis} \Rightarrow y = 0 \quad \lim_{(x,0) \to (0,0)} \frac{x^2}{x^2} = 1 \\ C_2: (x,y) \to (0,0) \text{ along } y\text{-axis} \Rightarrow x = 0 \quad \lim_{(0,y) \to (0,0)} \frac{-y^2}{y^2} = -1 \end{array}\right\} \text{So Limit DNE}$$

$\boxed{\text{Example 3}}$ Show that $\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4}$ DNE.

$$\left.\begin{array}{l} C_1: (x,y) \to (0,0) \text{ along } x\text{-axis} \Rightarrow y = 0 \quad \lim_{(x,0) \to (0,0)} \frac{0}{x^2} = 0 \\ C_2: (x,y) \to (0,0) \text{ along } y^2 = x \Rightarrow \lim_{(y^2,y) \to (0,0)} \frac{y^4}{y^4 + y^4} = \frac{1}{2} \end{array}\right\} \text{So limit DNE}$$

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Example 4 Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Try paths: $y=0$, $x=0$, $y=x$, $y=x^2$ all go to zero so lets try to prove

Let $\varepsilon>0$ We need to pick/find $\delta>0$ so that if $(x,y)\in D$

with $\sqrt{(x-0)^2+(y-0)^2}<\delta$ then $\underbrace{\left|\frac{3x^2y}{x^2+y^2}-0\right|<\varepsilon}_{\text{rearrange to find } x^2+y^2<?}$

$\frac{3x^2|y|}{x^2+y^2}<\varepsilon$ Since $x^2\le x^2+y^2 \Rightarrow \frac{x^2}{x^2+y^2}\le 1$ so $\frac{3x^2|y|}{x^2+y^2}\le 3|y|$

$3|y| = 3\sqrt{y^2}\le 3\sqrt{y^2+x^2} = 3\delta$ must be $<\varepsilon$ Choose $\delta=\varepsilon/3$

Proof: let $\varepsilon>0$ pick $\delta=\varepsilon/3$ then for all $(x,y)\in D$ with $\sqrt{(x^2+y^2)}<\delta$ we have

$$\left|\frac{3x^2y}{x^2+y^2}-0\right| = \frac{3x^2|y|}{x^2+y^2}\le 3|y|\le 3\sqrt{x^2+y^2} = 3\delta=\varepsilon \quad \blacksquare$$

- Continuous at (a,b): if $f(a,b)$ exists and $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

- Continuous on D: if f is continous at each point (a,b) in D.

Theorem $\lim_{(x,y)\to(a,b)} x = a$ $\quad \lim_{(x,y)\to(a,b)} y = b$ $\quad \lim_{(x,y)\to(a,b)} c = c$

Corollary All polynomials of two variables are continous.

Example 8 Is $f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0)\end{cases}$ Continous?

It's clear that $f(x,y)$ is continuous at all points but $(0,0)$

But by Ex. 4 $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0) \Rightarrow f(x,y)$ is continous on all $\mathbb{R}^2$.

- Extra Examples:

\#9 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-4y^2}{x^2+2y^2} = \boxed{DNE}$

C_1: $(x,y)\to(0,0)$ along x-axis $\Rightarrow y=0$ $\lim\limits_{(x,0)} \dfrac{x^4}{x^2} = 0$

C_2: $(x,y)\to(0,0)$ along $y=x \Rightarrow \lim\limits_{x\to 0} \dfrac{x^4-4x^2}{3x^2} = \lim\limits_{x\to 0} \frac{1}{3}x^2 - \frac{4}{3} = -\frac{4}{3}$

} limit DNE

\#13 $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = \boxed{0}$

Let $\varepsilon>0$ need $\delta>0$ with $\sqrt{x^2+y^2}<\delta$ then

$\left|\dfrac{xy}{\sqrt{x^2+y^2}} - 0\right| = \dfrac{|x||y|}{\sqrt{x^2+y^2}}$ since $|x| = \sqrt{x^2} \le \sqrt{x^2+y^2}$ then $\dfrac{|x|}{\sqrt{x^2+y^2}} \le 1$

So $\dfrac{|x||y|}{\sqrt{x^2+y^2}} \le |y| = \sqrt{y^2} \le \sqrt{y^2+x^2} < \delta = \varepsilon$ by picking $\delta=\varepsilon$ at start. ∎

\#39 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2} = \lim\limits_{r\to 0} \dfrac{r^3(\cos^3\theta+\sin^3\theta)}{r^2} = \lim\limits_{r\to 0} r(\cos^3\theta+\sin^3\theta) = \boxed{0}$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

} Can switch to Polar Coords but θ won't approach any value so only works if the limit is the same for all θ

\#40 $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

$$= \lim_{r\to 0} r^2\ln(r^2) = \lim_{r\to 0} \frac{\ln(r^2)}{r^{-2}} \overset{\text{L'Hopital}}{=} \lim_{r\to 0} \frac{\frac{1}{r^2}\cdot 2r}{-2r^{-3}} = \lim_{r\to 0} \frac{2}{r}\cdot\frac{r^3}{2}$$

$$= \lim_{r\to 0} r^2 = \boxed{0}$$