Section 13.4 - Motion in Space

We can use vector functions to describe particle motion:

- Position: $\vec{r}(t)$
- Velocity: $\vec{r}\,'(t) = \vec{v}(t)$
- acceleration: $\vec{r}\,''(t) = \vec{a}(t)$
- speed: $|\vec{v}(t)| = v$

- **Particle Motion is Cool:** youtube.com/watch?v=FG_11oacWbQ

  Each water droplet can be thought of as a particle, each only given acceleration due to gravity.

Example 3 A moving particle starts at an initial position $\vec{r}(0) = \langle 0,0,0 \rangle$ with initial velocity $\vec{v}(0) = \langle 1,-1,1 \rangle$. It's acceleration is $\vec{a}(t) = \langle 4t, 6t, 1 \rangle$. Find its velocity and position at time t.

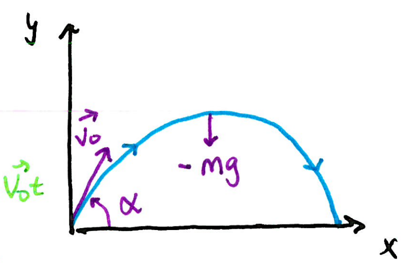
$$\vec{v}(t) = \int \vec{a}(t)\,dt$$
$$= \langle 2t^2, 3t^2, t \rangle + \vec{C}$$
$$\boxed{\vec{v}(t) = \langle 2t^2+1, 3t^2-1, t+1 \rangle}$$
$$\vec{C} = \langle 1,-1,1 \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt$$
$$= \langle \tfrac{2}{3}t^3 + t, t^3 - t, \tfrac{1}{2}t^2 + t \rangle + \vec{C}$$
$$\vec{C} = \langle 0,0,0 \rangle$$
$$\boxed{\vec{r}(t) = \langle \tfrac{2}{3}t^3 + t, t^3 - t, \tfrac{1}{2}t^2 + t \rangle}$$

- **Newton's Second Law of Motion:** $\vec{F}(t) = m\vec{a}(t)$

  $\vec{F}$ Force acting on an object of mass m produces acceleration $\vec{a}$

Example 5 A projectile is fired with an angle of elevation α and initial velocity $\vec{v}_0$. Assume air resistance is negligible and the only external force is gravity. Find $\vec{r}(t)$ and α that maximizes the horizontal range.



$$\vec{F} = -mg\,\vec{j} \Rightarrow \vec{a}(t) = \langle 0, -g \rangle$$

$$\vec{v}(t) = \langle 0, -gt \rangle + \vec{v}_0 \qquad \vec{r}(t) = \langle 0, -\tfrac{1}{2}gt^2 \rangle + \vec{v}_0 t$$

$$\vec{v}_0 = \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle \qquad \text{since } \vec{r}(0) = 0$$

$$\boxed{\vec{r}(t) = \langle (v_0 \cos\alpha)t, (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2 \rangle}$$

Horizontal Range: $y = 0$ $\quad t_{max} = \dfrac{2v_0 \sin\alpha}{g}$ $\quad x(t_{max}) = v_0 \cos\alpha \left(\dfrac{2v_0 \sin\alpha}{g} \right)$ $\quad \boxed{\text{Max at } \alpha = \frac{\pi}{4}}$

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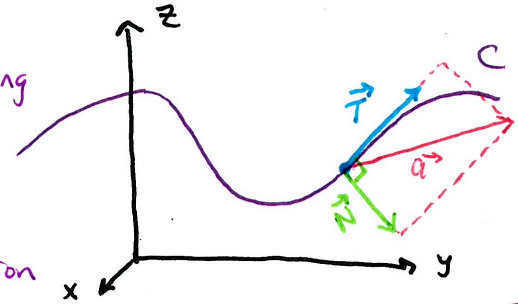
- Tangential & Normal Components of acceleration:



Rate of change in speed

Tangential Acceleration: acceleration acting in the direction of motion

Rate of change of velocity direction

Normal Acceleration: acceleration orthogonal to the tangential acceleration

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \frac{\vec{v}}{v} \quad \text{where } v = |\vec{v}| \quad \text{so} \quad \vec{v} = v\vec{T}$$

[velocity in terms of speed & unit tangent]

Differentiate both sides wrt t: $\vec{a} = v'\vec{T} + v\vec{T}\,'$

$$\vec{N} = \frac{\vec{T}\,'}{|\vec{T}\,'|} \quad \text{so} \quad \vec{T}\,' = \vec{N}\,|\vec{T}\,'| \quad \text{also} \quad K = \frac{|\vec{T}\,'|}{v} \quad \text{so} \quad \vec{T}\,' = \vec{N}vK$$

Thus $\boxed{\vec{a} = \underbrace{v'}_{\text{Tangential Component}}\vec{T} + \underbrace{v^2K}_{\text{Normal component}}\vec{N}}$

Example #36

(a) If a particle moves along a straight line, what can be said about its acceleration vector?

Moving in a straight line $\Rightarrow K = 0$

Thus $\boxed{\vec{a} = v'\vec{T}}$ acceleration only in direction of motion

(b) If a particle moves with constant speed along a curve, what can be said about its acceleration vector?

$v' = 0$ since v is constant

Thus $\boxed{\vec{a} = v^2K\vec{N}}$ acceleration only orthogonal to tangent vector

Section 13.4 – Motion in Space

MVC

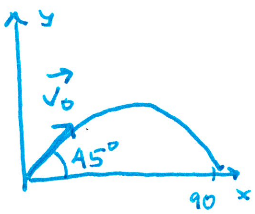
- Extra Examples

#22 Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

$C = |\vec{r}\,'(t)| = |\vec{v}(t)|$ WTS: $\vec{v}(t)\cdot\vec{a}(t) = \vec{r}\,'(t)\cdot\vec{r}\,''(t) = 0$

$C^2 = \vec{v}(t)\cdot\vec{v}(t)$ by Product Rule $0 = \frac{d}{dt}(\vec{v}(t)\cdot\vec{v}(t)) = 2\vec{v}(t)\cdot\vec{a}(t)$ ∎

#25 A ball is thrown at 45° to the ground. If the ball lands 90m away, what was the initial speed?



time t: $y = 0$ $x = 90$

① $t(V_0 \sin 45°) - \frac{1}{2}gt^2 = 0$

$t\left(V_0\frac{\sqrt{2}}{2} - \frac{g}{2}t\right) = 0$

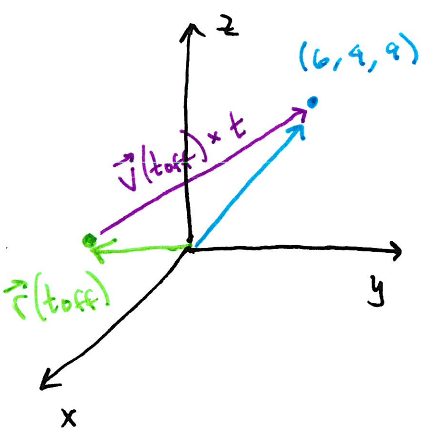
$t = \frac{V_0\sqrt{2}}{g}$

② $t(V_0\cos 45°) = 90$

$\frac{\sqrt{2}V_0^2}{g}\frac{\sqrt{2}}{2} = 90$

$\boxed{V_0 = \sqrt{90\cdot g}\ \text{m/s}}$

#45 The position of a spaceship is: $\vec{r}(t) = \left\langle (3+t), (2+\ln t), \left(7 - \frac{4}{t^2+1}\right)\right\rangle$ and the coordinates of a space station are $(6,4,9)$. The Captain wants the spaceship lined up with the space station so it can coast in with engines off. When should he turn off the engines?



Need:

$\vec{r}(t_{off}) + \vec{r}\,'(t_{off})\cdot t = \langle 6,4,9\rangle$

$\left\langle 3 + t_{off} + t,\ 2 + \ln t_{off} + \frac{t}{t_{off}},\ 7 - \frac{4}{t_{off}^2+1} + \frac{8t\,t_{off}}{(t_{off}^2+1)^2}\right\rangle = \langle 6,4,9\rangle$

① $3 + s + t = 6$ $s = t_{off}$ $\Rightarrow t = 3 - s$

② $2 + \ln s + \frac{t}{s} = 4$

③ $\frac{7(s^2+1)^2 - 4(s^2+1) + 8ts}{(s^2+1)^2} = 9$

$7(s^2+1)^2 - 4(s^2+1) + 8(3-s)s = 9(s^2+1)^2$

$\boxed{t_{off} = s = 1 \quad \text{so } t = 2}$