Section 13.3 – Arc Length & Curvature

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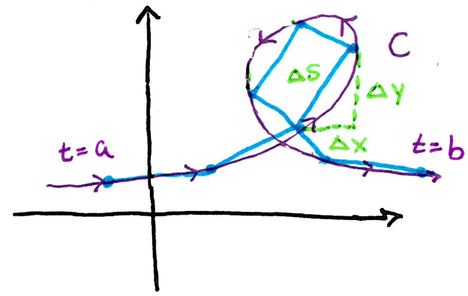
- Arc length of a curve C with parametric equations $x = f(t)$ and $y = g(t)$ (Section 10.2)

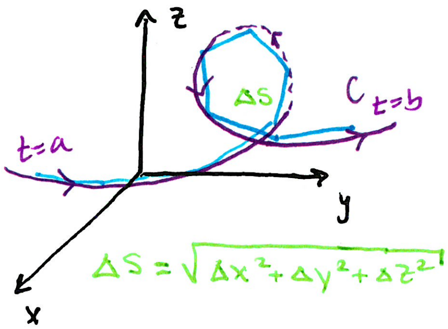
Approx. with Line-segment lengths:



$$L \approx \sum \Delta s = \sum \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sum \sqrt{f'(t)^2 + g'(t)^2}\,\Delta t$$

Do the same in 3D:



$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt = \int_a^b |\vec{r}\,'(t)|\,dt$$

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt = \int_a^b |\vec{r}\,'(t)|\,dt$$

Arc length function: $\boxed{s(t) = \int_a^t |\vec{r}\,'(u)|\,du}$

Example 1 Find the length of $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$ from $(1,0,0)$ to $(1,0,2\pi)$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1\rangle$ $\qquad t = 0$ to $t = 2\pi$

$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$$L = \int_0^{2\pi} \sqrt{2}\,dt = \boxed{\sqrt{2}\cdot 2\pi}$$

- Parametrize a curve with respect to Arc length:

Why? 1) arc length is more natural to the shape of C than t
2) arc length doesn't depend on choice of coord system

Idea: $s(t)$ is arc length as a function of t
$t(s) = s^{-1}$ is time as a function of s
$\Big\}\ \vec{r}(t) = \vec{r}(t(s)) = \vec{r}(s)$

Example 2 Reparametrize the helix $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$ wrt to arc length measured from $(1,0,0)$ in the direction of increasing t.

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}\,t \Rightarrow t(s) = \frac{s}{\sqrt{2}}$$

$$\boxed{\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right\rangle}$$

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- smooth parametrization $\vec r(t)$: if $\vec r\,'(t) \neq \vec 0$ and is continuous on $I \subseteq \mathbb{R}$.

- smooth curve C: if its parametrization $\vec r(t)$ is smooth.

  ★ Visually no sharp corners or cusps or vertical tangents

- Curvature: is the measure of how quickly the curve C is changing direction at a point on the curve.

★ Curvature of the Earth: thiscolossal.com/wp-content/uploads/2018/01/roads2.gif

$$\boxed{K = \left|\frac{d\vec T}{ds}\right| = \left|\frac{d\vec T/dt}{ds/dt}\right| = \frac{|\vec T\,'(t)|}{|\vec r\,'(t)|}}$$

How tangent vector changes wrt length

$S(t) = \int_0^t |\vec r\,'(u)|\,du$

so $S'(t) = |\vec r\,'(t)|$

Example 3 Show that the curvature of a circle of radius a is $1/a$.

$\vec r(t) = \langle a\cos t, a\sin t\rangle \qquad \vec r\,'(t) = \langle -a\sin t, a\cos t\rangle$

$|\vec r\,'(t)| = a \qquad \vec T(t) = \dfrac{\vec r\,'(t)}{|\vec r\,'(t)|} = \langle -\sin t, \cos t\rangle$

$\vec T\,'(t) = \langle -\cos t, -\sin t\rangle \qquad |\vec T\,'(t)| = 1$ thus $\boxed{K = \dfrac{|\vec T\,'(t)|}{|\vec r\,'(t)|} = \dfrac{1}{a}}$

Theorem

$$K(t) = \frac{|\vec r\,'(t) \times \vec r\,''(t)|}{|\vec r\,'(t)|^3}$$

Proof: $\vec T = \dfrac{\vec r\,'}{|\vec r\,'|} \qquad |\vec r\,'| = \dfrac{ds}{dt}$ so $\vec r\,' = \dfrac{ds}{dt}\vec T$

Differentiate wrt t using product rule

$$\vec r\,'' = \frac{d^2s}{dt^2}\vec T + \frac{ds}{dt}\vec T\,' \qquad \vec r\,' \times \vec r\,'' = \cancelto{\vec 0}{\vec r\,' \times \frac{d^2s}{dt^2}\frac{\vec r\,'}{|\vec r\,'|}} + \vec r\,' \times \frac{ds}{dt}\vec T\,'$$

$$|\vec r\,' \times \vec r\,''| = |\vec r\,' \times |\vec r\,'|\,\vec T\,'| = \big|\,|\vec r\,'|^2\, \vec T \times \vec T\,'\big|$$

$$= |\vec r\,'|^2\, \cancelto{1}{|\vec T|}\, |\vec T\,'| \sin\cancelto{1}{\pi/2} \qquad [\vec T \perp \vec T\,' \Rightarrow \theta = \pi/2]$$

Dividing both sides by $|\vec r\,'|^3$ gives:

$$\frac{|\vec r\,' \times \vec r\,''|}{|\vec r\,'|^3} = \frac{|\vec T\,'|}{|\vec r\,'|} = K(t)$$

Example 4 Find the curvature of $\vec r(t) = \langle t, t^2, t^3\rangle$ at $(0,0,0)$.

$\vec r\,'(t) = \langle 1, 2t, 3t^2\rangle \quad \vec r\,'(0) = \langle 1, 0, 0\rangle$

$\vec r\,''(t) = \langle 0, 2, 6t\rangle \quad \vec r\,''(0) = \langle 0, 2, 0\rangle$

$\vec r\,'(0) \times \vec r\,''(0) = \langle 0, 0, 2\rangle$

$$K(0) = \frac{|\vec r\,'(0) \times \vec r\,''(0)|}{|\vec r\,'(0)|} = \frac{2}{1} = \boxed{2}$$

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- Curvature of a function $y = f(x)$:

$\vec{r}(x) = \langle x, f(x), 0\rangle$

$\vec{r}\,'(x) = \langle 1, f'(x), 0\rangle$

$\vec{r}\,''(x) = \langle 0, f''(x), 0\rangle$

$\vec{r}\,'(x) \times \vec{r}\,''(x) = \langle 0, 0, f''(x)\rangle$

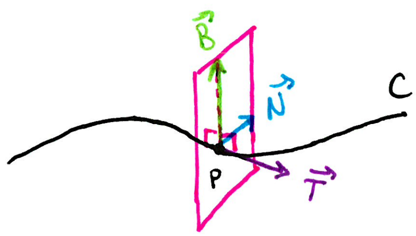
so $\boxed{\kappa(x) = \dfrac{|f''(x)|}{(1+f'(x)^2)^{3/2}}}$

- **Normal & Binormal vectors:** Note $\vec{T}\,' \perp \vec{T}$

Unit Normal vector: $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$ unit vector $\perp \vec{T}$

Binormal vector: $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ unit vector $\perp \vec{T}$ & $\vec{N}$

Normal plane to C at a point P: plane containing $\vec{N}$ & $\vec{B}$ and P



Normal vector to the Normal Plane is $\vec{T}(t)$ or $\vec{r}\,'(t)$!

$\boxed{\text{Example 7}}$ Find the equation of the normal plane to $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$ at $(0, 1, \pi/2)$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1\rangle$ at $t = \pi/2$

$\vec{r}\,'(\pi/2) = \langle -1, 0, 1\rangle$

Plane: $\boxed{-1(x-0) + 0(y-1) + 1(z-\pi/2) = 0}$ or $\boxed{-x+z=\pi/2}$

- Question: Why not talk about a tangent plane to C at point p?

Because the "tangent plane" would include the tangent line but there are an infinite number of planes that do that. Think "how would you lay a sheet of paper tangent to a string?"

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- Extra Examples

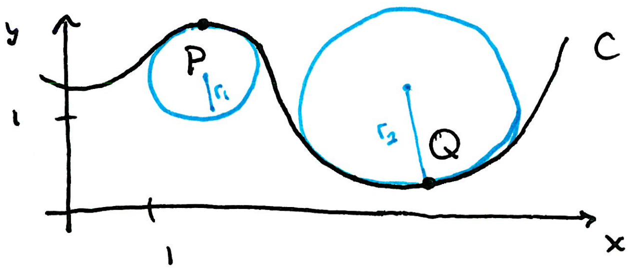
#31 At what point does $y = e^x$ have max curvature? What happens to the curvature as $x \to \infty$?

$$K(x) = \frac{e^x}{(1+e^{2x})^{3/2}} \qquad 0 = K'(x) = \frac{e^x(1+e^{2x})^{3/2} - e^x \frac{3}{2}(1+e^{2x})^{1/2} \cdot 2e^{2x}}{(1+e^{2x})^3} = \frac{(e^x)(1+e^{2x})^{1/2}\left[1-2e^{2x}\right]}{(1+e^{2x})^3}$$

when $1 - 2e^{2x} = 0$ so $\boxed{x = \frac{1}{2}\ln\left(\frac{1}{2}\right)}$ is a max since $K'(x)$ goes from + to −

$$\lim_{x\to\infty} K(x) = \lim_{x\to\infty} \frac{e^x}{e^{3x}} = \lim_{x\to\infty} e^{-2x} = \boxed{0}$$

#33 (a) Is the curvature at P or Q greater? Explain



$K(P) \approx \frac{1}{r_1}$

$K(Q) \approx \frac{1}{r_2}$

$\boxed{K(P) > K(Q) \text{ since } r_1 < r_2}$

#46 Consider the curvature of the family of functions $y = e^{cx}$ at $x = 0$. For which members is $K(0)$ largest?

$$K(x) = \frac{c^2 e^{cx}}{(1+c^2e^{2cx})^{3/2}} \qquad K(0) = \frac{c^2}{(1+c^2)^{3/2}} \qquad 0 = K'(c) = \frac{2c(1+c^2)^{3/2} - c^2(1+c^2)^{1/2}\cdot\frac{3}{2}(2c)}{(1+c^2)^{3/2}}$$

$$= \frac{c(1+c^2)^{1/2}\left[2-c^2\right]}{(1+c^2)^{3/2}}$$

when $2 - c^2 = 0$ so $c = \pm\sqrt{2}$

sign of $K'(c)$

−   +   −

$-\sqrt{2}$ $\sqrt{2}$

$\boxed{K(0) \text{ is largest at } c = \sqrt{2}}$

#53 At what point on the curve $x = t^3, y = 3t, z = t^4$ is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

$$\vec{n} = \langle 6, 6, -8 \rangle \sim \vec{r}\,'(t) = \langle 3t^2, 3, 4t^3 \rangle$$

1) $6k = 3t^2$ $3 = 3t^2 \Rightarrow t = \pm 1$

2) $6k = 3 \Rightarrow k = \frac{1}{2}$

3) $-8k = 4t^3$ Check $-4 = 4t^3 \Rightarrow t = -1$

$\boxed{\text{Point}: (-1, -3, 1)}$