Section 13.2 - Derivatives & Integrals of Vector Functions

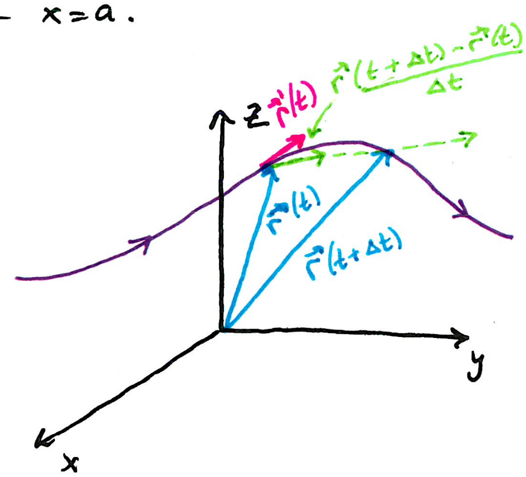
Recall: The derivative of a function $y=f(x)$ at a point $x=a$ represents the slope of the line tangent to $f(x)$ at $x=a$.

- First approx. slope between two points $(a, f(a))$ and $(a+h, f(a+h))$

$$\frac{f(a+h)-f(a)}{h}$$

- Limit as $h \to 0$ gave the slope at one point $x=a$ i.e. slope of tangent line at $x=a$.

- Tangent Vector (Derivative of $\vec{r}(t)$) $\vec{r}\,'(t)$:



$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$ provided the limit exists

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$ is the Unit Tangent vector

Theorem If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ and f, g, h are differentiable, then

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Proof: $$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t)-f(t)}{\Delta t}, \frac{g(t+\Delta t)-g(t)}{\Delta t}, \frac{h(t+\Delta t)-h(t)}{\Delta t} \right\rangle$$

$$= \langle f'(t), g'(t), h'(t) \rangle$$ since f, g, h are differentiable. ∎

Example 1 (a) Find the derivative of $\vec{r}(t) = \langle (1+t^3), te^{-t}, \sin 2t \rangle$

(b) Find the unit tangent vector to $\vec{r}(t)$ when $t=0$.

(a) $$\vec{r}\,'(t) = \langle \underbrace{3t^2}_{\text{Power rule}}, \underbrace{\tfrac{d}{dt}(t)\cdot e^{-t} + t\tfrac{d}{dt}(e^{-t})}_{\text{Product rule}}, \underbrace{\cos(2t)\cdot \tfrac{d}{dt}(2t)}_{\text{Chain rule}} \rangle = \boxed{\langle 3t^2, e^{-t} - te^{-t}, 2\cos 2t \rangle}$$

(b) $$\vec{T}(0) = \frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \frac{\langle 0, 1, 2 \rangle}{\sqrt{5}} = \boxed{\langle 0, \sqrt{5}/5, 2\sqrt{5}/5 \rangle}$$

Example 3 Find parametric equations for the tangent line to the helix:
$x = 2\cos t \quad y = \sin t \quad z = t$ at $(0, 1, \pi/2)$

$\vec{r}\,'(t) = \langle -2\sin t, \cos t, 1 \rangle$ at $(0,1,\pi/2)$ $t = z = \pi/2$

$\vec{r}\,'(\pi/2) = \langle -2, 0, 1 \rangle \leftarrow$ Direction of tangent line point: $(0, 1, \pi/2)$

$$\boxed{x = 0 - 2t \quad y = 1 + 0t \quad z = \pi/2 + t}$$

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- **Differentiation Rules**: $\vec{u}, \vec{v}$ differentiable, c a scalar, f differentiable

1. $\frac{d}{dt}(\vec{u}(t)+\vec{v}(t)) = \vec{u}\,'(t)+\vec{v}\,'(t)$
2. $\frac{d}{dt}(c\vec{u}(t)) = c\vec{u}\,'(t)$
3. $\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$
4. $\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$
5. $\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$
6. $\frac{d}{dt}(\vec{u}(f(t))) = \vec{u}\,'(f(t))\,f'(t)$ [Chain Rule]

Product Rules

Example 4 Show that if $|\vec{r}(t)| = c$ then $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$.

Show: $\vec{r}\,'(t)\cdot\vec{r}(t) = 0$. Know: $\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = c^2$

Then differentiating wrt t gives: [wrt: with respect to]

$$0 = \frac{d}{dt}(\vec{r}(t)\cdot\vec{r}(t)) = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) = 2\vec{r}\,'(t)\cdot\vec{r}(t)$$

So $0 = \vec{r}\,'(t)\cdot\vec{r}(t)$ ∎

- **Definite Integral**: $\int_a^b \vec{r}(t)\,dt = \lim_{n\to\infty}\sum_{i=1}^{n}\vec{r}(t_i)\Delta t = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$

- **Indefinite Integral**: $\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$

Note: Constant of integration is now a vector!

* FTC: $\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$ where $\vec{R}\,'(t) = \vec{r}(t)$

Example 5 If $\vec{r}(t) = \langle 2\cos t, \sin t, 2t\rangle$ find $\int \vec{r}(t)\,dt$ and $\int_0^{\pi/2}\vec{r}(t)\,dt$?

$$\int \vec{r}(t)\,dt = \langle 2\sin t + C_1, -\cos t + C_2, t^2 + C_3\rangle = \boxed{\langle 2\sin t, -\cos t, t^2\rangle + \vec{C}}$$

Write this way

$$\int_0^{\pi/2}\vec{r}(t)\,dt = \left\langle 2\sin\tfrac{\pi}{2}, -\cos\tfrac{\pi}{2}, \left(\tfrac{\pi}{2}\right)^2\right\rangle - \langle 2\sin 0, -\cos(0), 0^2\rangle = \boxed{\left\langle 2, 1, \tfrac{\pi^2}{4}\right\rangle}$$

Question: What does $\int_a^b \vec{r}(t)\,dt$ represent?

Vector of net change in each component

useful to find the average value: $\frac{1}{b-a}\int_a^b \vec{r}(t)\,dt$

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- Extra Examples

#27 Find a vector equation for the tangent line to the curve of intersection of:

$$x^2+y^2=25 \quad \text{and} \quad y^2+z^2=20 \quad \text{at} \quad (3,4,2)$$

$x = 5\cos t$ $\qquad \frac{dx}{dt} = -5\sin t$ $\qquad \left.\frac{dx}{dt}\right|_{(3,4,2)} = -5(4/5) = -4$

$y = 5\sin t$ $\qquad \frac{dy}{dt} = 5\cos t$ $\qquad \left.\frac{dy}{dt}\right|_{(3,4,2)} = 5(3/5) = 3$

$z = \sqrt{20-25\sin^2 t}$ $\qquad \frac{dz}{dt} = \frac{1}{2}(20-25\sin^2 t)^{-1/2}\cdot(-50\sin t\cdot\cos t)$ $\qquad \frac{dz}{dz} = \frac{1}{2}\left(20-25(4/5)^2\right)^{-1/2}\left(-50(4/5)(3/5)\right) = -6$

$\begin{cases} \cos t = 3/5 \\ \sin t = 4/5 \end{cases}$

$$\boxed{\vec{r}(t) = \langle 3-4t,\ 4+3t,\ 2-6t \rangle}$$

#33 $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\vec{r}_2 = \langle \sin t, \sin 2t, t \rangle$ both intersect the origin, Find their angle of intersection.

$\vec{r}_1{}'(t) = \langle 1, 2t, 3t^2 \rangle$ $\qquad \vec{r}_2{}'(t) = \langle \cos t, 2\cos 2t, 1 \rangle$

$\vec{r}_1{}'(0) = \langle 1, 0, 0 \rangle$ $\qquad \vec{r}_2{}'(0) = \langle 1, 2, 1 \rangle$

$$\cos\theta = \frac{\vec{r}_1{}'(0)\cdot\vec{r}_2{}'(0)}{|\vec{r}_1{}'(0)|\,|\vec{r}_2{}'(0)|} = \frac{1}{\sqrt{6}} \qquad \boxed{\theta \approx 65.91^\circ}$$

#40 $$\int\left(te^{2t}\vec{i} + \frac{t}{1-t}\vec{j} + \frac{1}{\sqrt{1-t^2}}\vec{k}\right)dt = \left\langle \int te^{2t}dt,\ \int\frac{t}{1-t}dt,\ \int\frac{1}{\sqrt{1-t^2}}dt \right\rangle$$

$$\int te^{2t}dt = t\left(\tfrac{1}{2}e^{2t}\right) - \int \tfrac{1}{2}e^{2t}dt = \boxed{\tfrac{1}{2}te^{2t} - \tfrac{1}{4}e^{2t} + C_1}$$

Integrate by parts $u=t$ $dv=e^{2t}$

$$\int \frac{t}{1-t}dt = \frac{-t(1-t)^{-2}}{2} + \int (1-t)^{-2}dt = \boxed{\frac{-t}{2(1-t)^2} - \frac{1}{3(1-t)^3} + C_2}$$

Integrate by parts $u=t$ $dv=(1-t)^{-1}$

$$\int \frac{1}{\sqrt{1-t^2}}dt = \boxed{\arcsin(t) + C_3}$$

#53. If $\vec{r}(t) \neq \vec{0}$ show that $\frac{d}{dt}|\vec{r}(t)| = \frac{1}{|\vec{r}(t)|}\vec{r}(t)\cdot\vec{r}\,'(t)$. Hint: $|\vec{r}(t)|^2 = \vec{r}(t)\cdot\vec{r}(t)$

$$\frac{d}{dt}|\vec{r}(t)|^2 = 2|\vec{r}(t)|\cdot\frac{d}{dt}|\vec{r}(t)|$$

$$\|$$

$$\frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = 2\vec{r}\,'(t)\cdot\vec{r}(t)$$

$$\Rightarrow \frac{d}{dt}|\vec{r}(t)| = \frac{\vec{r}\,'(t)\cdot\vec{r}(t)}{|\vec{r}(t)|} \quad \blacksquare$$