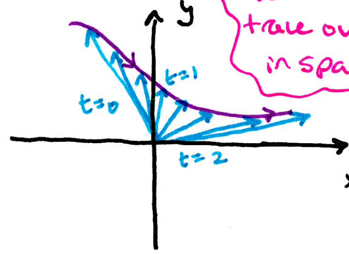
Section 13.1 - Vector Functions & Space Curves

MVC

Vector functions trace out a curve in space



- **Vector Functions:** a function with domain a set of $\mathbb{R}$ and whose range is a set of V^n

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

- $\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$

Provided each component's limit exists otherwise we say it DNE.

- $\vec{r}(t)$ is continuous at $t=a$ if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$ and $\vec{r}(a)$ exists.

Example 2 Find $\lim_{t \to 0} \vec{r}(t)$ where $\vec{r}(t) = (1+t^3)\vec{i} + te^{-t}\vec{j} + \frac{\sin t}{t}\vec{k}$

ALL limit rules still apply

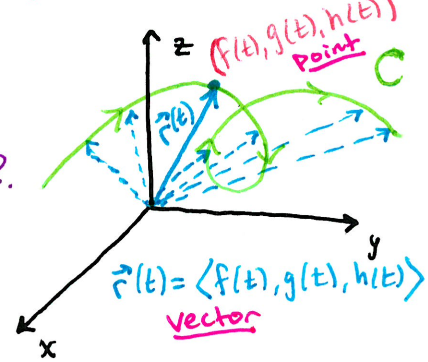
$\lim_{t \to 0} \vec{r}(t) = \boxed{\langle 1, 0, 1 \rangle}$ since

$\lim_{t \to 0} (1+t^3) = 1+(0)^3 = 1$	$\lim_{t \to 0} \frac{t}{e^t} = \frac{0}{e^0} = 0$	$\lim_{t \to 0} \frac{\sin t}{t} = \lim_{t \to 0} \frac{\cos t}{1} = \frac{\cos(0)}{1} = 1$
since $(1+t^3)$ is continuous	since t/e^t is continuous	by L'Hopital's Rule

- A **space curve** C is the set of points (x, y, z) where

Parametric Equations – $\left[x = f(t) \quad y = g(t) \quad z = h(t) \right]$

as t varries throughout an interval $I \subseteq \mathbb{R}$.



$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$
vector

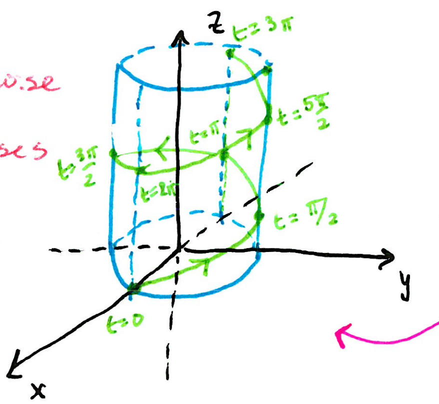
Example 4 Sketch the curve whose vector equation is $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$$

$x = \cos t$, $y = \sin t$ } Counter clockwise Circle as t increases

$z = t$ — goes up as t increases

$x^2 + y^2 = 1$
↓
Curve lies on Cylinder



Counter Clockwise spiraling upwards about z-axis as t increases.

This Curve is called a **helix**

Question: How can Ex. 4 be changed to spiral clockwise?

$x = \cos t \qquad y = -\sin t \qquad z = t$

Example 5 Find a vector equation and parametric equations for the line segment that joins the point $P(1,3,-2)$ to $Q(2,-1,3)$.

$$\vec{PQ} = \langle 2-1, -1-3, 3+2 \rangle = \langle 1, -4, 5 \rangle$$

$$\vec{r}(t) = \langle 1, 3, -2 \rangle + \langle 1, -4, 5 \rangle \cdot t$$

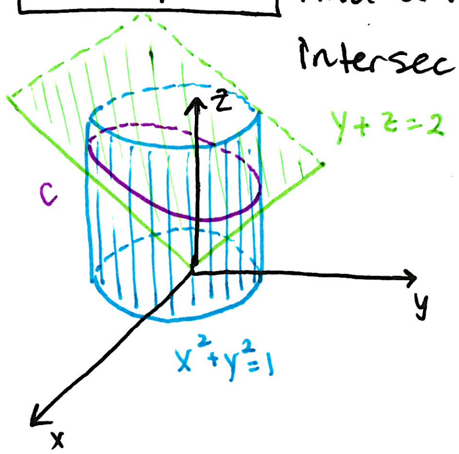
$$\boxed{\vec{r}(t) = \langle 1+t,\ 3-4t,\ -2+5t \rangle \quad 0 \le t \le 1}$$

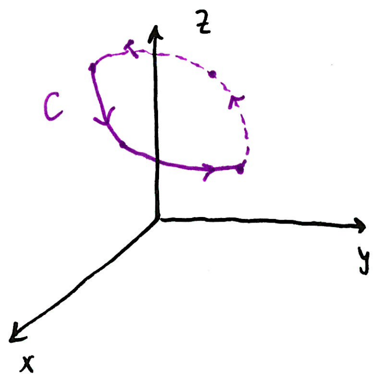
Vector

Parametric

$$\boxed{\begin{aligned} x &= 1+t \\ y &= 3-4t \\ z &= -2+5t \\ 0 &\le t \le 1 \end{aligned}}$$

Example 6 Find a vector function that represents the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $y+z=2$.





$$x^2+y^2=1 \ \begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

$$z = 2-y = 2-\sin t$$

$$\boxed{\vec{r}(t) = \langle \cos t, \sin t, 2-\sin t \rangle \quad 0 \le t \le 2\pi}$$

One time around C

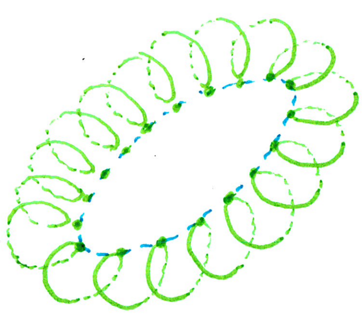
Example Use a computer to graph the Toroidal Spiral

$$x = (2+\sin 20t)\cos t \qquad y = (4+\sin 20t)\sin t \qquad z = \cos 20t$$

→ Wolframalpha.com

→ math.uri.edu/~bkaskosz/flashmo/parcur/

Top View:

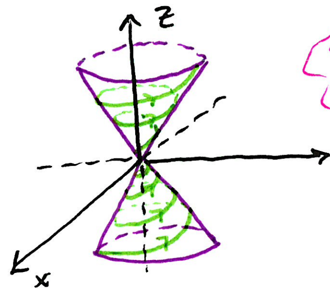


Think of an extended Slinky connected end to end in a circle/ellipse.

- Extra Examples

#27 Show that the curve: $x = t\cos t$, $y = t\sin t$, $z = t$ lies on the cone $z^2 = x^2 + y^2$ and use that to help sketch the curve.

$$\underbrace{x^2 + y^2 = (t\cos t)^2 + (t\sin t)^2}_{\text{Counterclockwise circles of increasing radius as } |t| \text{ increases}} = t^2 = z^2$$



Spiral that gets fitted into a cone

#41 Find the vector function that represent the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$.

$$(1+y)^2 = \left(\sqrt{x^2+y^2}\right)^2 \;;\; z \geq 0$$

$$1 + 2y + y^2 = x^2 + y^2$$

$$1 + 2y = x^2$$

$$y = \frac{x^2-1}{2} \qquad z = 1 + \frac{x^2-1}{2} = \frac{x^2+1}{2}$$

$$\boxed{\vec{r}(t) = \left\langle t, \frac{t^2-1}{2}, \frac{t^2+1}{2} \right\rangle}$$

#48. Two particles travel along the space curves:

$$\vec{r}_1(t) = \langle t, t^2, t^3 \rangle \quad \text{and} \quad \vec{r}_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle$$

Do the particles collide? Do their paths intersect?

Collide: $\vec{r}_1(t) = \vec{r}_2(t)$

$t = 1 + 2t \Rightarrow t = -1$

$t^2 = 1 + 6t$ but $(-1)^2 \neq 1 + 6(-1)$

$t^3 = 1 + 14t$ $\quad(-1)^3 \neq 1 + 14(-1)$

$\boxed{\text{No will not Collide}}$

Intersect: $\vec{r}_1(t) = \vec{r}_2(s)$

1) $t = 1 + 2s$
2) $t^2 = 1 + 6s$
3) $t^3 = 1 + 14s$

$(1+2s)^2 = 1 + 6s$

$-2s + 4s^2 = 0$

$s = 0, \tfrac{1}{2}$

1) $s = 0, \tfrac{1}{2}$

$t = 1, 2$

3) check:

$(1)^3 = 1 + 14(0)$ ✓

$(2)^3 = 1 + 14(\tfrac{1}{2})$ ✓

$\boxed{\text{Yes paths will intersect at } (1,1,1) \text{ and } (2,4,8)}$

★ Cool: Watch video on space filling curves: youtube.com/watch?v=RU0wScIj360