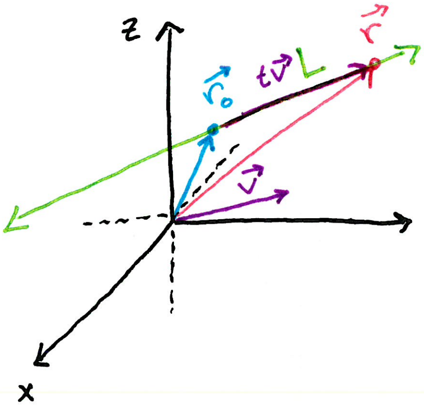
Section 12.5 – Equations of Lines & Planes

- Line in 2D: Need a point and a slope (direction)
- Line in 3D: Need a point (r_0) and a direction ($\vec{v}$) (vector)



- Vector Equation: $\boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$
- Parametric Equations: $\vec{v} = \langle a, b, c \rangle$ $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

$$\boxed{x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct}$$

- Symmetric Equations: eliminate parameter

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

Example 2 (a) Find the parametric equations of the line through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.
(b) At what point does the line intersect the xy-plane?

(a) $\vec{r}_0 = \langle 2, 4, -3 \rangle$ $\vec{v} = \langle 3-2, -1-4, 1+3 \rangle = \langle 1, -5, 4 \rangle$

$$\boxed{\begin{aligned} x &= 2 + t \\ y &= 4 - 5t \\ z &= -3 + 4t \end{aligned}}$$

(b) $z = 0$ so $t = 3/4$

$x = \frac{11}{4}$ $y = \frac{1}{4}$

$$\boxed{\left(\frac{11}{4}, \frac{1}{4}, 0\right)}$$

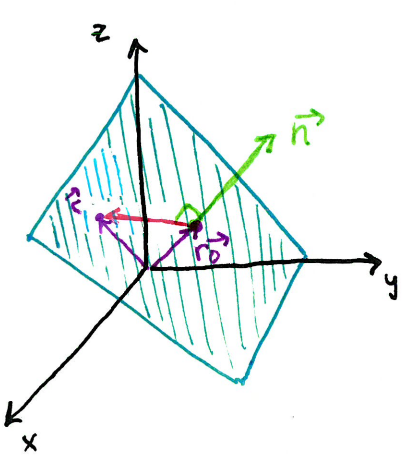
- Line Segment: between points r_0 and r_1

$$\vec{r}(t) = \vec{r}_0 + (\vec{r}_1 - \vec{r}_0)t = (1-t)\vec{r}_0 + t\vec{r}_1 \ ; \quad 0 \le t \le 1$$

- Skew Lines: Do not intersect and are not parallel
- Planes in 3D: Need a point and a normal vector (perpendicular to plane)

$\vec{n}$: normal vector

r, r_0: points on plane



- Vector Equation: $\boxed{\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0}$
- Scalar Equation: $\vec{n} = \langle a, b, c \rangle$ $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ $\vec{r} = \vec{x}$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

- Linear Equation: $d = -ax_0 - by_0 - cz_0$

$$\boxed{ax + by + cz + d = 0}$$

Example 5 Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$.

$\vec{a} = \vec{PQ} = \langle 2,-4,4\rangle$

$\vec{b} = \vec{PR} = \langle 4,-1,-2\rangle$

$$\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \langle 12, 20, 14\rangle$$

$$\boxed{12(x-1) + 20(y-3) + 14(z-2) = 0}$$

or $\boxed{6x + 10y + 7z - 50 = 0}$

Example 7 Find the angle between the planes:

$$x + y + z = 1 \quad \text{and} \quad x - 2y + 3z = 1$$

$\vec{n}_1 = \langle 1,1,1\rangle$

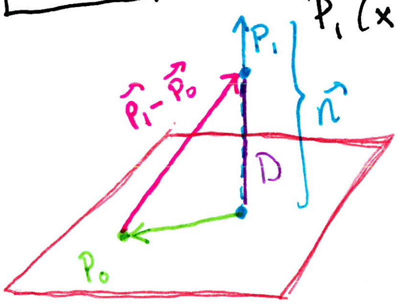
$\vec{n}_2 = \langle 1,-2,3\rangle$

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{1-2+3}{\sqrt{3}\sqrt{14}} = \frac{2}{\sqrt{42}} \qquad \boxed{\theta \approx 72.025^\circ}$$

- Question: How can you determine if two planes are parallel without finding the angle between them?

  ★ If the plane's normal vectors are scalar multiples of one another.

Example 8 Find a formula for the distance D from a point $P_1(x_1,y_1,z_1)$ to the plane $ax + by + cz + d = 0$.



$P_0(x_0,y_0,z_0)$ any point on plane

$\vec{P_1} - \vec{P_0} = \langle x_1-x_0,\ y_1-y_0,\ z_1-z_0\rangle$

$\vec{n} = \langle a,b,c\rangle$ normal vector

$D = |\text{Comp}_{\vec{n}}\,\vec{P_1}-\vec{P_0}|$ Scalar project $\vec{P_1}-\vec{P_0}$ onto $\vec{n}$

$$= \frac{|\vec{n}\cdot(\vec{P_1}-\vec{P_0})|}{|\vec{n}|} = \boxed{\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$$

Since $d = -ax_0 - by_0 - cz_0$

- CPM 3D Plotter: technology.cpm.org/general/3dgraph/ Graph: $x+y+z=1$, $3x+y-z=2$

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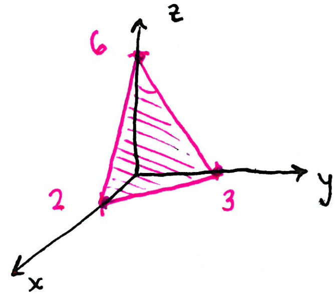
MVC

- Extra Examples:



#42 sketch $3x + y + 2z = 6$ using intercepts

x-intercept: $y = z = 0 \quad x = 2$

y-intercept: $x = z = 0 \quad y = 6$

z-intercept: $x = y = 0 \quad z = 3$

#48 where does the line through $(1,0,1)$ and $(4,-2,2)$ intersect the plane $x + y + z = 6$.

$X = 1 + (4-1)t \qquad y = 0 + (-2-0)t \qquad z = 1 + (2-1)t$

$x = 1 + 3t \qquad y = -2t \qquad z = 1 + t$

$6 = (1+3t) - 2t + (1+t) \Rightarrow 4 = 2t \Rightarrow t = 2$ $\boxed{\text{Point: } (7,-4,3)}$

#52 Determine if the planes are parallel, perpendicular or neither.

$$2z = 4y - x \qquad \text{and} \qquad 3x - 12y + 6z = 1$$

$\vec{n}_1 = \langle -1, 4, -2 \rangle \qquad \vec{n}_2 = \langle 3, -12, 6 \rangle$

$\vec{n}_2 = -3\,\vec{n}_1$ thus the planes are $\boxed{\text{parallel}}$

#63. Find an equation of the plane with x-intercept a, y-intercept b, and z-intercept c.

x-intercept: $y = z = 0 \quad x = a \quad \Rightarrow \quad \frac{x}{a} + 0 + 0 = 1$

y-intercept: $x = z = 0 \quad y = b \quad \Rightarrow \quad \frac{x}{a} + \frac{y}{b} + 0 = 1$

z-intercept: $x = y = 0 \quad z = c \quad \Rightarrow \quad \boxed{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1}$

#75. Show that the distance between parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $D = \dfrac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

From Ex. 8 $\quad D = \dfrac{|ax_1 + by_1 + cz_1 + d_1|}{\sqrt{a^2+b^2+c^2}}$ where $P_1(x_1, y_1, z_1)$ is a point on the second plane

$ax_1 + by_1 + cz_1 = -d_2$ so $\boxed{D = \dfrac{|-d_2 + d_1|}{\sqrt{a^2+b^2+c^2}}}$