Section 12.4 - The Cross Product

Recall: dot product of two vectors produced a scalar
Would like a product that is meaningful and produces a vector

- Cross Product: $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$

Only defined in V^3!

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Better way to compute: use the Determinant of a 3x3 matrix

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Example 1 Compute $\vec{a} \times \vec{b}$ for $\vec{a} = \langle 1, 3, 4 \rangle$, $\vec{b} = \langle 2, 7, -5 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \vec{i} \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} = (3(-5) - 4 \cdot 7)\vec{i} - (1(-5) - 4 \cdot 2)\vec{j} + (7 \cdot 1 - 3 \cdot 2)\vec{k} = \boxed{\langle -43, 13, 1 \rangle}$$

Theorem The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Proof: $(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle$

$$= a_1 a_2 b_3 - a_1 a_3 b_2 + a_2 a_3 b_1 - a_1 a_2 b_3 + a_1 a_3 b_2 - a_2 a_3 b_1 = 0 \quad \blacksquare$$

Theorem If θ is the angle between $\vec{a}$ and $\vec{b}$ then $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$.

Proof: $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta) = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$ $\blacksquare$

write componentwise and rearrange

use $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$\sin^2\theta + \cos^2\theta = 1$

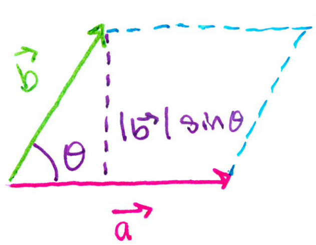
Corollary Two nonzero vectors $\vec{a}, \vec{b}$ are parallel iff $\vec{a} \times \vec{b} = \vec{0}$.

Proof: $(\Rightarrow)$ $\vec{a} \parallel \vec{b}$ then $\theta = \pi, 0$ so $\sin\theta = 0$ thus $|\vec{a} \times \vec{b}| = 0$ so $\vec{a} \times \vec{b} = \vec{0}$

$(\Leftarrow)$ $\vec{a} \times \vec{b} = \vec{0}$, $|\vec{a}|, |\vec{b}| \neq 0$ then $\sin\theta = 0$ so $\theta = \pi, 0$ hence $\vec{a} \parallel \vec{b}$. $\blacksquare$

Questions: ① What does $|\vec{a} \times \vec{b}|$ represent geometrically in relation to $\vec{a}$ and $\vec{b}$?

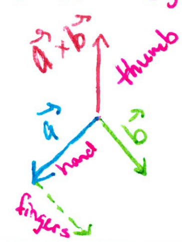
② Do you think the cross product is commutative: $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$?

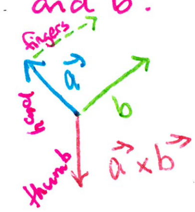


① $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ - same as area of the parallelogram formed by $\vec{a}$ and $\vec{b}$.

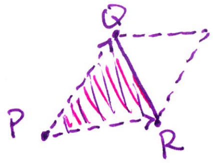
② using definition $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Direction of cross product vector given by Right hand Rule





Example 4 Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.



$\vec{PQ} = \langle -3, 1, -7 \rangle$

$\vec{PR} = \langle 0, -5, -5 \rangle$

Area of $\Delta PQR = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} |\langle -40, -15, 15 \rangle|$

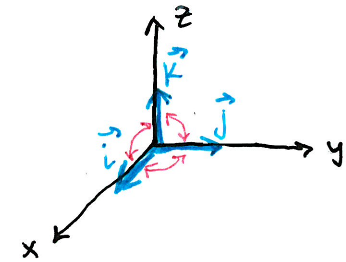
$= \boxed{\frac{5}{2}\sqrt{82}}$

Warning:

- Compute using the Right Hand Rule:

$\vec{i} \times \vec{j} = \vec{k}$ $\quad$ $\vec{j} \times \vec{k} = \vec{i}$ $\quad$ $\vec{i} \times \vec{k} = -\vec{j}$

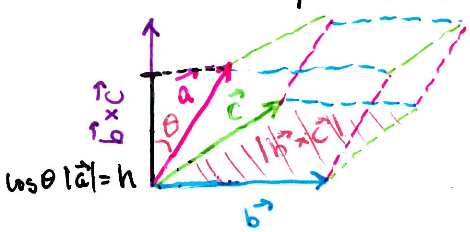
$\vec{j} \times \vec{i} = -\vec{k}$ $\quad$ $\vec{k} \times \vec{j} = -\vec{i}$ $\quad$ $\vec{k} \times \vec{i} = \vec{j}$



Theorem

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. ★ $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

★ Scalar Triple Product: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ Scalar value



Geometrically:

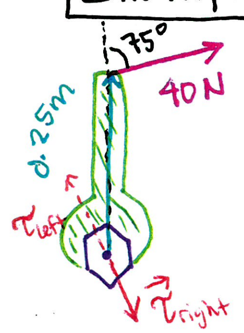
$\vec{a} \cdot (\vec{b} \times \vec{c}) = |\vec{a}||\vec{b} \times \vec{c}| \cos\theta = \underbrace{|\vec{b} \times \vec{c}|}_{\text{area of base}} \times \underbrace{|\vec{a}| \cos\theta}_{\text{height}}$ $\boxed{\text{Volume of a parallelpipe}}$

Example 5 Use the scalar triple product to show $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$, $\vec{c} = \langle 0, -9, 18 \rangle$ are coplanar (all in the same plane).

Means Volume of Parallelpiped is 0: $\begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1(-18 - 4(-9)) - 4(36 - 0) - 7(-18 - 0) = \boxed{0}$

- Application: Torque ($\vec{\tau}$) - measuring the tendency of the body to rotate about the origin, when a force ($\vec{F}$) acts on the rigid body. (position $\vec{r}$)

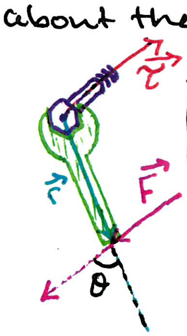
Example 6 A bolt is tightened by applying 40N of force to a 0.25m wrench. Find the magnitude of the torque about the bolt center.



$|\tau| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}| \sin\theta$

$= (0.25\,m)(40N) \sin 75°$

$\approx \boxed{9.66\, Nm}$

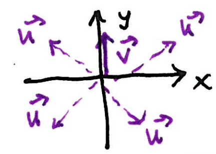


$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$

Section 12.4 – The Cross Product

• Extra Examples:

\# 42. Let $\vec{v} = 5\vec{j}$ and let $\vec{u}$ be a vector with length 3 that starts at the origin and rotates in the xy plane. Find the max and min values of $|\vec{u}\times\vec{v}|$. In what direction does $\vec{u}\times\vec{v}$ point?



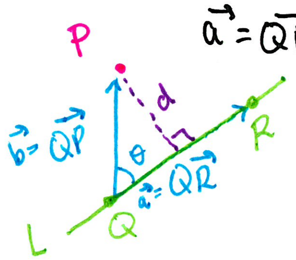
$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta = 3\cdot 5\sin\theta \quad , \quad 0 \le \theta \le \pi$

$|\vec{u}\times\vec{v}|$ is a max when $\sin\theta$ is a max at $\pi/2$ so max of $\boxed{|\vec{u}\times\vec{v}| = 15}$

$|\vec{u}\times\vec{v}|$ is a min when $\sin\theta$ is a min at $0, \pi$ so min of $\boxed{|\vec{u}\times\vec{v}| = 0}$

$\boxed{\vec{u}\times\vec{v} \sim \vec{k}}$ when $\vec{u}$ in Q2, Q3 $\quad \boxed{\vec{u}\times\vec{v} \sim -\vec{k}}$ when $\vec{u}$ in Q1, Q4

\# 45(a). Let P be a point not on the line L, passing through points Q and R. Show that the distance d from P to L is $d = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}|}$ where $\vec{a} = \vec{QR}$ and $\vec{b} = \vec{QP}$.



$$d = |\vec{b}|\sin\theta = \cancel{|\vec{b}|}\cdot\frac{|\vec{a}\times\vec{b}|}{\cancel{|\vec{b}|}|\vec{a}|} = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|} \quad \blacksquare$$

\# 48. If $\vec{a}+\vec{b}+\vec{c} = \vec{0}$ show that $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$.

$(\vec{a}+\vec{b}+\vec{c})\times\vec{b} = \vec{0}\times\vec{b}$

$\vec{a}\times\vec{b} + \cancel{\vec{b}\times\vec{b}}^{\,\vec{0}} + \vec{c}\times\vec{b} = \vec{0}$

$\underline{\vec{a}\times\vec{b}} = -\vec{c}\times\vec{b} = \underline{\vec{b}\times\vec{c}}$

$(\vec{a}+\vec{b}+\vec{c})\times\vec{a} = \vec{0}\times\vec{a}$

$\cancel{\vec{a}\times\vec{a}}^{\,\vec{0}} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a} = \vec{0}$

$\underline{\vec{c}\times\vec{a}} = -\vec{b}\times\vec{a} = \underline{\vec{a}\times\vec{b}} \quad \blacksquare$

\# 53. Suppose that $\vec{a} \ne \vec{0}$.

(a) If $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ does it follow $\vec{b} = \vec{c}$? $\boxed{\text{No}}$

$\vec{a} = \langle 1, 0\rangle \quad \vec{b} = \langle 0, 1\rangle \quad \vec{c} = \langle 0, 0\rangle \quad \vec{a}\cdot\vec{b} = 0 = \vec{a}\cdot\vec{c}$

(b) If $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$ does it follow $\vec{b} = \vec{c}$? $\boxed{\text{No}}$

$\vec{a} = \langle 1, 0, 0\rangle \quad \vec{b} = \langle 0, 1, 0\rangle \quad \vec{a}\times\vec{b} = \langle 0, 0, 1\rangle$

$\vec{c} = \langle 1, 1, 0\rangle \quad \vec{a}\times\vec{c} = \langle 0, 0, 1\rangle$

(c) If $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ and $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$ does it follow $\vec{b} = \vec{c}$? $\boxed{\text{Yes}}$

① $|\vec{b}|\cos\theta_1 = |\vec{c}|\cos\theta_2$ ② $|\vec{b}|\sin\theta_1 = |\vec{c}|\sin\theta_2$

②/① : $\tan\theta_1 = \tan\theta_2$ for $0 \le \theta_1, \theta_2 < 180 \Rightarrow \underbrace{\theta_1 = \theta_2}_{\text{same direction}}$ by ① $\underbrace{|\vec{b}| = |\vec{c}|}_{\text{same magnitude}}$