Section 12.3 - The Dot Product

Q: Is it possibly to "multiply" two vectors with the result being meaningful?

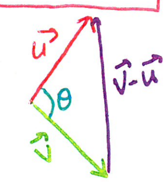
- **Dot Product** (Scalar Product): $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Multiply Componentwise, Add

$\boxed{\text{Result is always a Scalar}}$

$\boxed{\text{Theorem}}$ If θ is the angle between $\vec{v}$ and $\vec{u}$ then $\vec{v} \cdot \vec{u} = |\vec{v}||\vec{u}|\cos\theta$



Proof: Apply Law of Cosines to the triangle formed by $\vec{u}$ and $\vec{v}$

$$|\vec{v} - \vec{u}|^2 = |\vec{v}|^2 + |\vec{u}|^2 - 2|\vec{v}||\vec{u}|\cos\theta$$

in V^2: $(v_1 - u_1)^2 + (v_2 - u_2)^2 = v_1^2 + v_2^2 + u_1^2 + u_2^2 - 2|\vec{u}||\vec{v}|\cos\theta$

$$-2v_1u_1 - 2v_2u_2 = -2|\vec{u}||\vec{v}|\cos\theta$$

$$\underbrace{v_1u_1 + v_2u_2}_{} = |\vec{u}||\vec{v}|\cos\theta$$

Meaningful quantity why we define the Dot product

$$\vec{v} \cdot \vec{u} = |\vec{u}||\vec{v}|\cos\theta \quad \blacksquare$$

- **Properties of Dot Product**:

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

vector Scalar

$\boxed{\text{Corollary}}$ $\cos\theta = \dfrac{\vec{v} \cdot \vec{u}}{|\vec{v}||\vec{u}|}$ Rearrange above Theorem

$\boxed{\text{Corollary}}$ $\vec{v}$ and $\vec{u}$ are orthogonal (perpendicular) if and only if $\vec{v} \cdot \vec{u} = 0$ with $\vec{v}, \vec{u} \neq \vec{0}$.

Proof: If $\vec{v} \cdot \vec{u} = 0$ with $\vec{v}, \vec{u} \neq 0$ then by Theorem above $\cos\theta = 0 \Rightarrow \theta = \pi/2$

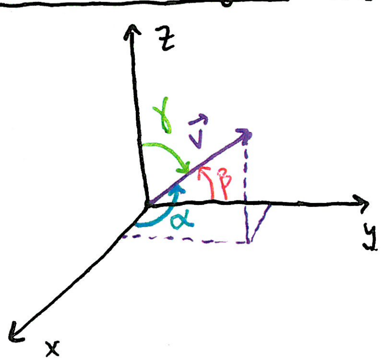
If $\theta = \pi/2 \Rightarrow \cos\theta = 0 \Rightarrow \vec{v} \cdot \vec{u} = 0$. $\blacksquare$

$\boxed{\text{Example 3}}$ Find the angle between $\vec{a} = \langle 2, 2, -1 \rangle$ and $\vec{b} = \langle 5, -3, 2 \rangle$

$|\vec{a}| = 3$ $\quad |\vec{b}| = \sqrt{38}$ $\quad \vec{a} \cdot \vec{b} = 2(5) + 2(-3) - 1(2) = 2$

$$\cos\theta = \frac{2}{3\sqrt{38}} \Rightarrow \theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx \boxed{84^\circ}$$

- **Directional Angles**: The angles $\vec{v}$ makes with the positive x, y, z axes



$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$ since unit basis vectors

$$\cos\alpha = \frac{\vec{a} \cdot \vec{i}}{|\vec{a}||\vec{i}|} = \frac{a_1}{|\vec{a}|} \qquad \cos\beta = \frac{\vec{a} \cdot \vec{j}}{|\vec{a}||\vec{j}|} = \frac{a_2}{|\vec{a}|} \qquad \cos\gamma = \frac{\vec{a} \cdot \vec{k}}{|\vec{a}||\vec{k}|} = \frac{a_3}{|\vec{a}|}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_1^2 + a_2^2 + a_3^2}{|\vec{a}|^2} = 1$$

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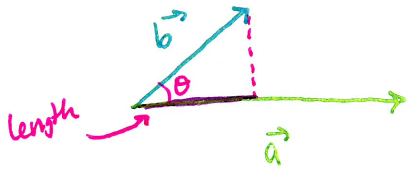
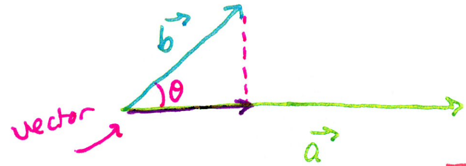
- **Projections:**

Scalar Projection of $\vec{b}$ onto $\vec{a}$



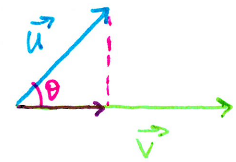
Vector Projection of $\vec{b}$ onto $\vec{a}$



$$\text{Comp}_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta = \boxed{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}} = |\text{Proj}_{\vec{a}}\vec{b}|$$

$$\text{Proj}_{\vec{a}}\vec{b} = \text{Comp}_{\vec{a}}\vec{b}\,\frac{\vec{a}}{|\vec{a}|} = \boxed{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}}$$

Example 6 Find the scalar and vector projections of $\vec{u} = \langle 1,1,2\rangle$ onto $\vec{v} = \langle -2,3,1\rangle$

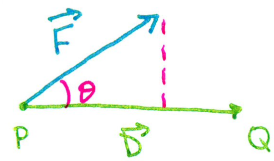


$$\text{Comp}_{\vec{v}}\vec{u} = \frac{\vec{v}\cdot\vec{u}}{|\vec{v}|} = \frac{-2+3+2}{\sqrt{14}} = \boxed{\frac{3}{\sqrt{14}}}$$

$$\text{Proj}_{\vec{v}}\vec{u} = \left(\text{Comp}_{\vec{v}}\vec{u}\right)\frac{\vec{v}}{|\vec{v}|} = \frac{3}{\sqrt{14}}\,\frac{\vec{v}}{\sqrt{14}} = \boxed{\left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle}$$

- **Applications:** Work – force constant in direction of displacement $W = F\cdot|\vec{D}|$

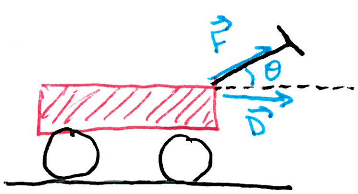
★ Constant Force not in direction of $\vec{D}$ given by $\vec{F}$



Work $W = |\vec{F}|\cos\theta\,|\vec{D}| = \vec{F}\cdot\vec{D}$

done in direction of $\vec{D}$

Example A wagon is pulled a distance of 100m along a horizontal path by a constant force of 70N. The handle is held at some angle above the horizon. If the work done was about 5734 J find the angle of the handle above the horizon.



$|\vec{D}| = 100m$ $\quad |\vec{F}| = 70N$

$W = 5734J$

$$5734 = 100\cdot 70N\cdot\cos\theta$$

$$\cos\theta = \frac{5734}{100\cdot 70}$$

$$\boxed{\theta \approx 35^\circ}$$

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- Extra Examples:

① If $\vec{a} = \langle 1, 2, 3 \rangle$ find $\vec{b}$ so that $\text{comp}_{\vec{a}}\, \vec{b} = 3$.

$\vec{b} = \langle b_1, b_2, b_3 \rangle$ $\quad 3 = \text{Comp}_{\vec{a}}\, \vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \dfrac{b_1 + 2b_2 + 3b_3}{\sqrt{14}}$

So $\quad 3\sqrt{14} = b_1 + 2b_2 + 3b_3$ $\quad$ Choose any b_1, b_2, b_3 that works : $\boxed{b = \langle 0, 0, \sqrt{14} \rangle}$

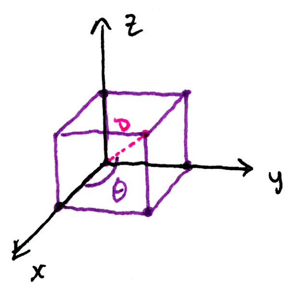
#48. Suppose $\vec{a}$ and $\vec{b}$ are nonzero vectors. When is $\text{Comp}_{\vec{a}}\, \vec{b} = \text{Comp}_{\vec{b}}\, \vec{a}$?

$\text{Comp}_{\vec{a}}\, \vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

$\text{Comp}_{\vec{a}}\, \vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$\boxed{\text{Equal if } |\vec{a}| = |\vec{b}| \text{ that is when the vectors have the same length.}}$

55. Find the angle between a diagonal of a cube and one of its edges.



$\vec{D} = \langle 1, 1, 1 \rangle$

$\vec{\text{Edge}}$: $\langle 1, 0, 0 \rangle$ or $\langle 0, 1, 0 \rangle$ or $\langle 0, 0, 1 \rangle$

$\cos\theta = \dfrac{\vec{D} \cdot \vec{\text{Edge}}}{|\vec{D}||\vec{\text{Edge}}|} = \dfrac{1}{\sqrt{3}}$

$\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right) \approx \boxed{54.74°}$

61. Use Theorem 3 to prove Cauchy-Schwarz Inequality: $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$

Proof:

$$\boxed{\begin{aligned} |\vec{a} \cdot \vec{b}| &= \big||\vec{a}||\vec{b}|\cos\theta\big| && \text{by Theorem 3} \\ &= |\vec{a}||\vec{b}||\cos\theta| && \text{Since } |\vec{a}|, |\vec{b}| \geq 0 \\ &\leq |\vec{a}||\vec{b}| \cdot 1 && \text{Since } |\cos\theta| \leq 1 \end{aligned}}$$ ∎