# Structure of the Socle and Radical Series for Projective Indecomposable Modules of Simple Groups

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# Overview

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# Overview of Project

- Looked at the structure of the socle and radical series for projective indecomposable modules of simple groups
- Not a lot of work and developed theory in this field
- Generated experimental data to develop theortical questions and conjectures

# Motivation

- Classify *FG*-modules: construct all possible *FG*-modules.
  - First idea: Look at the composition series of a module whose factors are simple modules. Lots of work done here but hard to build the modules.
  - Second idea: Find all projective indecomposable modules and form quotients of direct sums of projective indecomposable modules (PIMs).

# Background

Let $G$ be a finite group and $F$ a field of characteristic $p$.

**Theorem**

*(**Maschke's Theorem**) A $FG$-module is semisimple if and only if the characteristic of $F$ does not divide the order of $G$.*

- Interested in what happens when $p \mid |G|$
- Want to measure the closeness to being semisimple

# Background

## Definition

A $R$-module $M$ is **indecomposable** if $M \neq 0$ and $M$ cannot be written as a direct sum of two non-zero submodules.

## Definition

A $R$-module $P$ is **projective** if to every surjective homomorphism $f : M \to N$ of $R$-modules and to every homomorphism $g : P \to N$ there exists a homomorphism $h : P \to M$ with $f \circ h = g$.

- Projective indecomposable modules are precisely the direct summands of $FG$.

# Background

**Definition**

The **socle series** for a module $M$ is an increasing chain of submodules defined by $s_1(M) = \text{soc}(M)$ and

$$s_i(M)/s_{i-1}(M) = \text{soc}(M/s_{i-1}(M)).$$

**Definition**

The **radical series** for a module $M$ is a decreasing chain of submodules defined by $r_1(M) = \text{rad}(M)$ and

$$r_i = \text{rad}(r_{i-1}(M)),$$

where $M/rad(M)$ is the *head* of $M$.

# Background

## Theorem

*The radical series terminates in 0 in the $k^{th}$ step if and only if the socle series terminates in $M$ in the $k^{th}$ step.*

# Background

**Theorem**

*The radical series terminates in 0 in the $k^{th}$ step if and only if the socle series terminates in $M$ in the $k^{th}$ step.*

**Theorem**

*For $P$ a projective indecomposable $FG$-module, $\mathrm{soc}(P) \simeq P/\mathrm{rad}(P)$ is a simple $FG$-module. Every simple $FG$-module is isomorphic to $\mathrm{soc}(P) \simeq P/\mathrm{rad}(P)$ for some PIM $P$.*

- For $P$ a PIM, $\quad s_1(P) = r_{k-1}(P) \quad$ and $\quad r_1(P) = s_{k-1}(P)$
- What about the intermediate factors? When are the series equal?

# Background

- A PIM $P$ is upper-stable (lower-stable ) if

$$r_2(P) = s_{k-2}(P) \qquad\qquad (s_2(P) = r_{k-2}(P))$$

**Theorem**

*(**Landrock**) If all projective indecomposable FG-modules are either all upper or all lower stable then all PIMs have the same Loewy length.*

# Method

- Used GAP and package Basic to compute the group algebra, database of simple modules, PIMs, and the socle and radical series for PIMs
- Used matrices to keep track of the simple modules in each factor for easy comparison of the series

# Example

- Mathieu group $G = M_{11}$ and $F$ a field of characteristic 11
- 8 PIMs, 2 having differing socle and radical series
- Database 5 simple modules $a, b, c, d, e$

$$M_r(P_5) = \begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & 1 & \cdot & 1 & 1 \\ \cdot & 1 & 1 & 1 & 2 \\ \cdot & 1 & 1 & 1 & 3 \end{pmatrix} \end{array} \begin{matrix} r_5 \\ r_4 \\ r_3 \\ r_2 \\ r_1 \end{matrix} \qquad M_s(P_5) = \begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 2 \\ \cdot & 1 & \cdot & 1 & 2 \\ \cdot & 1 & 1 & 1 & 2 \\ \cdot & 1 & 1 & 1 & 3 \end{pmatrix} \end{array} \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{matrix}$$

# Results

- Computed series for about 100 group algebras
- Only about 30 had differing series

| Group | $Sz(8)$ | $Sz(8)$ | $L_2(17)$ | $M_{11}$ | $L_2(8)$ | $M_{11}$ | $U_3(3)$ | $M_{11}$ | $U_3(3)$ | $L_3(2)$ | $U_3(3)$ | $A_7$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Prime | 2 | 5 | 3 | 11 | 3 | 2 | 2 | 3 | 3 | 2 | 7 | 2 |
| Mult. | 6 | 1 | 2 | 1 | 2 | 4 | 5 | 2 | 3 | 3 | 1 | 3 |
| PIMs | 7 | 4 | 2 | 5 | 2 | 3 | 3 | 7 | 8 | 3 | 3 | 3 |
| DPs | 7 | 1 | 1 | 4 | 1 | 1 | 2 | 6 | 8 | 2 | 1 | 2 |
| TF | 27 | 4 | 5 | 5 | 5 | 5 | 19 | 7 | 9 | 5 | 5 | 5 |
| PSNTF | T | F | F | F | F | T | T | T | T | F | F | T |
| PMDF | 16 | 1 | 2 | 2 | 2 | 2 | 14 | 2 | 6 | 2 | 2 | 2 |
| PSNDF | T | T | T | T | T | T | F | T | F | T | T | T |
| PMDS | 7 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 9 | 1 | 1 | 1 |
| PSNDS | F | T | T | T | T | T | F | T | F | T | T | T |

# Results

| Group | $U_4(2)$ | $L_2(31)$ | $M_{12}$ | $U_4(2)$ | $L_2(23)$ | $L_3(3)$ | $M_{12}$ | $L_3(3)$ | $A_8$ | $L_3(3)$ | $L_2(32)$ | $L_3(4)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Prime | 2 | 2 | 2 | 3 | 2 | 13 | 3 | 2 | 2 | 3 | 11 | 2 |
| Mults. | 6 | 5 | 6 | 4 | 3 | 1 | 3 | 4 | 6 | 3 | 1 | 6 |
| PIMs | 7 | 3 | 3 | 5 | 3 | 3 | 8 | 3 | 7 | 8 | 2 | 5 |
| DPs | 7 | 2 | 3 | 3 | 2 | 1 | 8 | 2 | 7 | 8 | 1 | 5 |
| TF | 13 | 17 | 10 | 13 | 5 | 5 | 9 | 5 | 13 | 9 | 6 | 13 |
| PSNTF | T | F | T | F | F | F | T | T | T | T | F | T |
| PMDF | 10 | 14 | 5 | 10 | 2 | 2 | 6 | 2 | 10 | 6 | 3 | 8 |
| PSNDF | F | T | T | T | T | T | T | T | T | F | T | T |
| PMDS | 25 | 1 | 6 | 24 | 1 | 1 | 12 | 1 | 19 | 9 | 1 | 24 |
| PSNDS | F | T | F | F | T | T | F | T | F | F | T | F |

# Further Questions

- When do we have equality of the series for all PIMs?
- Relations when $p^n \mid |G|$ exactly?
- Bounds on number of differing factors?
- Bound on number of differing simples in factors?

# References


1. Carlson, J. F. *Modules and Group Algebras*. Basel: Birkhauser Verlag, 1996. Print.
2. Grove, Larry C. *Groups and Characters*. New York: Wiley, 1997. Print.
3. Hilton, Peter and Urs Stammbach. *A Course in Homological Algebra*. New York: Springer-Verlag, 1971. Print.
4. Hoffman, Tom and Klaus Lux. "Basic Package Documentation." June 2014.
5. James, G.D. "The Modular Characters of the Mathieu Groups." *Journal of Algebra* 27.1 (1973): 57-111. Web.
6. Landrock, Peter. "Some Remarks on Loewy Lengths of Projective Modules." *Lecture Notes in Mathematics* 832 (1980): 369-81. Web.
7. Serganova, Vera. "Representation Theory." Web. 1 Dec. 2014. https://math.berkeley.edu/ serganov/math252/notes9.pdf.

The End

Questions?