Pre-Calc AB

# Matrices

3/3/16
- 3/4/16

MATRIX Topics: Lessons 101, 105, 108
- 3x3 Determinants
- Solutions of 3x3 systems
- 0 Determinant / Independent Systems
- Matrix Multiplication
- Matrices in Calculator

3/3 : Handout Matrix WS I
Test back after Lesson

3/4 : Handout Matrix WS II

Recall: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

For 3x3

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

OR

$$\begin{matrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{matrix}$$

$$-\left[a_3 b_2 c_1 + a_1 c_2 b_3 + b_1 a_2 c_3\right]$$

$$+\left[a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3\right]$$

Ex. $\begin{vmatrix} 3 & -2 & 1 \\ -1 & 4 & 10 \\ 2 & -3 & 5 \end{vmatrix} = (3)\begin{vmatrix} 4 & 10 \\ -3 & 5 \end{vmatrix} - (-2)\begin{vmatrix} -1 & 10 \\ 2 & 5 \end{vmatrix} + (1)\begin{vmatrix} -1 & 4 \\ 2 & -3 \end{vmatrix}$

$= 3(20 - (-30)) + 2(-5 - 20) + (3 - 8)$

$= 150 + (-50) - 5 = \boxed{95}$

OR

$$\begin{matrix} 3 & -2 & 1 & 3 & -2 \\ -1 & 4 & 10 & -1 & 4 \\ 2 & -3 & 5 & 2 & -3 \end{matrix}$$

$$-\left[8 + (-90) + 10\right]$$

$$+\left[60 + (-40) + 3\right]$$

$\begin{vmatrix} 3 & -2 & 1 \\ -1 & 4 & 10 \\ 2 & -3 & 5 \end{vmatrix} = (60 - 40 + 3) - (8 - 90 + 10)$

$= \boxed{95}$

## Cramer's Rule:

$a_1 x + b_1 y + c_1 z = k_1$

$a_2 x + b_2 y + c_2 z = k_2$

$a_3 x + b_3 y + c_3 z = k_3$

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Ex. Use Cramer's Rule to Solve for $z$:

$$\begin{cases} 3x + 2y + z = 9 \\ \quad\;\; 2y + 3z = 14 \\ x + 2y \qquad\;\; = 3 \end{cases}$$

$$z = \frac{\begin{vmatrix} 3 & 2 & 9 \\ 0 & 2 & 14 \\ 1 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 0 \end{vmatrix}}$$

$$= \frac{3\begin{vmatrix} 2 & 14 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & 14 \\ 1 & 3 \end{vmatrix} + 9\begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix}}{3\begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix}} = \frac{3(6-28) - 2(-14) + 9(-2)}{3(-6) - 2(-3) + (-2)} = \frac{-66 + 28 - 18}{-18 + 6 - 2} = \frac{-56}{-14} = \boxed{4}$$

## Independent Systems:

A $2\times 2$ system is independent if the terms with variables in one equation are not multiples of the terms with variables in the other.

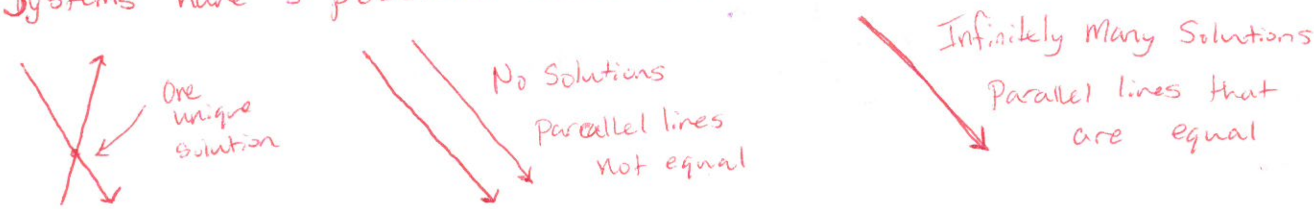
That is there is a unique solution to the equation, lines are not parallel.

Quick Way to Determine Independence:

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \text{ is independent } \iff \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

☆ Extends to $3\times 3$ and even $n \times n$ matrices

2x2 Systems have 3 possible Outcomes:




Ex: Determine if the following Systems are independent:

① $\begin{cases} 4x + 3y = 1 \\ 8x + 6y = 42 \end{cases}$

② $\begin{cases} 4x + 3y = 1 \\ 8x + 6y = 2 \end{cases}$ First × 2

$$\begin{vmatrix} 4 & 3 \\ 8 & 6 \end{vmatrix} = 24 - 24 = 0 \Rightarrow \text{Both are not independent}$$

① Lines are parallel not equal $\Rightarrow$ $\boxed{\text{No Solutions}}$

② Lines are parallel equal $\Rightarrow$ $\boxed{\text{Infinitely Many Solutions}}$

# Matrix Multiplication

- Size of a matrix, called the <u>order</u>, is the number rows × number of columns.

Ex. $A = \begin{bmatrix} 2 & 0 \\ 3 & 4 \\ 1 & 5 \end{bmatrix}$ is a $3 \times 2$ matrix because it has 3 rows and 2 columns.

- Matrix Multiplication of $A \cdot B$ is defined only when the number of the columns of A equals the number of rows of B.

$$\underset{(n \times m) \cdot (m \times p)}{A \cdot B} = \underset{(n \times p)}{C}$$

Ex. $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

| $A \cdot B$ | $B \cdot A$ | $B \cdot C$ | $C \cdot B$ |
|---|---|---|---|
| Not Defined | Defined | Not Defined | Not Defined |
| $(2 \times 2) \cdot (3 \times 2)$ | $(3 \times 2) \cdot (2 \times 2)$ $= (3 \times 2)$ | $(3 \times 2) \cdot (3 \times 1)$ | $(3 \times 1) \cdot (3 \times 2)$ |

$$B \cdot A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2+2 & 0+6 \\ 0+2 & 0+3 \\ 6+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 2 & 3 \\ 6 & 0 \end{bmatrix}$$

Examples:

$$\underset{(2 \times 2)}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \cdot \underset{(2 \times 2)}{\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix}} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$\underset{(1 \times 2)}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \cdot \underset{(2 \times 1)}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} = \underset{1 \times 1}{\begin{bmatrix} 0 \cdot 2 + 1 \cdot 3 \end{bmatrix}} = \begin{bmatrix} 3 \end{bmatrix}$$

$$\underset{(2 \times 1)}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} \cdot \underset{(1 \times 2)}{\begin{bmatrix} 0 & 1 \end{bmatrix}} = \underset{(2 \times 2)}{\begin{bmatrix} 2 \cdot 0 & 2 \cdot 1 \\ 3 \cdot 0 & 3 \cdot 1 \end{bmatrix}} = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -2 \\ 3 & 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 1(-1) + 2 \cdot 0 & 0 \cdot 0 + 1 \cdot 0 + 2 \cdot 1 & 0 \cdot 1 + 1(-2) + 2(3) \\ -1(1) + 0(-1) + -2(0) & 0 + 0 + -2 & -1 + 0 - 6 \\ 3 - 4 + 0 & 0 + 0 + 0 & 3 - 8 + 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 4 \\ -1 & -2 & -7 \\ -1 & 0 & -5 \end{bmatrix}$$